Analytical Methods and Perturbation Theory for the Elliptic Restricted Three-Body Problem of Astrodynamics

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Problem Statement

The elliptic restricted three-body problem (ERTBP) of astrodynamics considers the motion of a massless object (spacecraft) within the gravitational potential field generated by a non-circular two-body system (planet-moon).
Motivation I

While the general three-body problem (TBP) and circular restricted three-body problem (CRTBP) have been the focus of countless studies, comparatively little research has been conducted with regard to the ERTBP. This is due in part to the complicated dynamical nature of the ERTBP.
Motivation II

According to the latest census by the IAU Minor Planet Center there are over 5,000 identified objects librating about the Sun-Jupiter triangular Lagrange points and more still at those of Neptune, Mars and as of last year, even Earth.

Many spacecraft missions have also been proposed to the Earth-Moon and Sun-Earth triangular Lagrange points in order to study...
Outline

1. Elliptic Restricted Three-Body Problem
   - Spacecraft Dynamics
   - Hamiltonian Formulation

2. Canonical Perturbation Theory
   - Deprit-Hori Lie Transform Method for Two-Parameter Non-Autonomous Systems
   - DH Transformation of the ERTBP

3. Dynamical Analysis
   - Stability Analysis
   - Local Integrals of Motion
   - Control

4. Conclusions
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Elliptic Restricted Three-Body Problem

- Spacecraft Dynamics
ERTBP - Two-Body Problem

The gravitational two-body problem has a closed-form solution given by a conic section

\[ R(\nu) = \frac{p}{1 + e \cos \nu} \]

Closed orbit = circle or ellipse \((0 \leq e < 1)\)

The instantaneous position is defined by the angular true anomaly \(\nu\) whose time rate of change is

\[ \frac{d\nu}{dt} = \frac{\sqrt{G(m_1 + m_2)}}{p}(1 + e \cos \nu)^2 \]

Figure: Two Body Keplerian System.
In canonical units, \( m_1 + m_2 = 1 \), \( G = 1 \) and the orbital period of the primaries is \( 2\pi \).

The mass ratio is defined as

\[
\mu \triangleq \frac{m_2}{m_1 + m_2} = m_2
\]

such that \( 1 - \mu = m_1 \) and

\[
R_1 = \mu R \\
R_2 = (1 - \mu) R
\]

where

\[
R(\nu) = \frac{p}{1 + e \cos \nu}
\]
Define the $\hat{x}$-axis along the line connecting the two primaries with positive direction toward the larger primary.

Figure: Synod Barycentric Ref Frame
ERTBP - Coordinate System

The spacecraft position vector is expressed in Cartesian \((x, y, z)\) or Spherical \((r, \phi, \theta)\) coordinates:

\[
\vec{r} = x \hat{x} + y \hat{y} + z \hat{z} = r (\cos \theta \sin \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \phi \hat{z})
\]

The distances to the primaries are

\[
r_1^2 = (x - \mu R)^2 + y^2 + z^2 = r^2 + \mu^2 R^2 - 2 \mu r R \sin \phi \cos \theta
\]

\[
r_2^2 = (x + (1 - \mu) R)^2 + y^2 + z^2 = r^2 + (1 - \mu)^2 R^2 + 2 (1 - \mu) r R \sin \phi \cos \theta
\]
ERTBP - Spacecraft Inertial Energy

Specific Kinetic energy:

\[ 2T = \dot{v}^2 \]

\[ = (\dot{x} - \dot{v} y)^2 + (\dot{y} + \dot{v} x)^2 + \dot{z}^2 \]

\[ = r^2 + r^2 \dot{\phi}^2 + r^2 (\dot{v} + \dot{\theta})^2 \sin^2 \phi \]

Potential function:

\[ U = -\frac{1}{r_1} - \frac{\mu}{r_2} \]

The spacecraft dynamics are

\[ \frac{d\vec{v}}{dt} = -\nabla U(\vec{r}, t) \]
Elliptic Restricted Three-Body Problem
- Hamiltonian Formulation
ERTBP - Hamiltonian Formulation

The Lagrangian function is defined as the difference in the kinetic energy and the potential function \( \mathcal{L} = T - U \)

\[
\mathcal{L} = \frac{1}{2} \left[ \dot{r}^2 + r^2 \dot{\phi}^2 + r^2 \left( \dot{\nu} + \dot{\theta} \right)^2 \sin^2 \phi \right] + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}
\]

The generalized momenta \( p_i \) are defined by

\[
\begin{align*}
    p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r} & \dot{r} &= p_r \\
    p_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \dot{\phi} & \dot{\phi} &= \frac{p_\phi}{r^2} \\
    p_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2 \left( \dot{\nu} + \dot{\theta} \right) \sin^2 \phi & \dot{\theta} &= \frac{p_\theta}{r^2 \sin^2 \phi} - \dot{\nu}
\end{align*}
\]
ERTBP - Hamiltonian Function

The Hamiltonian function is defined as

\[ \mathcal{H}(q, p, t) = \sum_{i=1}^{n} (p_i \dot{q}_i) - \mathcal{L}(q, \dot{q}, t) = p_r \dot{r} + p_\phi \dot{\phi} + p_\theta \dot{\theta} - \mathcal{L}(q, \dot{q}, t) \]

\[ = \frac{1}{2} \left[ p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi} \right] - \dot{\nu} p_\theta - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} \]

\[ r_1^2 = r^2 + \mu^2 R^2 - 2\mu rR \sin \phi \cos \theta \]

\[ r_2^2 = r^2 + (1 - \mu)^2 R^2 + 2 (1 - \mu) rR \sin \phi \cos \theta \]

The Hamiltonian equations of motion are

\[ \frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i} \quad \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} \]
ERTBP - Hamiltonian Function

The Hamiltonian function is defined as

\[ \mathcal{H}(q, p, t) = \sum_{i=1}^{n} (p_i \dot{q}_i) - \mathcal{L}(q, \dot{q}, t) = p_r \dot{r} + p_\phi \dot{\phi} + p_\theta \dot{\theta} - \mathcal{L}(q, \dot{q}, t) \]

\[ = \frac{1}{2} \left[ p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi} \right] - \nu p_\theta - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} \]

\[ r_1^2 = r^2 + \mu^2 R^2 - 2 \mu r R \sin \phi \cos \theta \]

\[ r_2^2 = r^2 + (1 - \mu)^2 R^2 + 2 (1 - \mu) r R \sin \phi \cos \theta \]

The Hamiltonian equations of motion are

\[ \frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i} \quad \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} \]
The transformation $q = Q(\hat{q}, \hat{p}, t)$ and $p = P(\hat{q}, \hat{p}, t)$ is implicitly defined through a generating function $\mathcal{W}$ which depends on both the new and old variables. For example, one possible form is $\mathcal{W}(q, \hat{p}, t)$, which defines the transformation equations

$$p = \frac{\partial}{\partial q} \mathcal{W}(q, \hat{p}, t) \rightarrow P(q, \hat{p}, t) = P(Q(\hat{q}, \hat{p}, t), \hat{p}, t)$$

$$\hat{q} = \frac{\partial}{\partial \hat{p}} \mathcal{W}(q, \hat{p}, t) \rightarrow \hat{Q}(q, \hat{p}, t) = \hat{Q}(q, \hat{P}(q, p, t), t)$$

The transformation is canonical if the equations of motion may be represented in terms of a new Hamiltonian function $\mathcal{K}(\hat{q}, \hat{p}, t)$ that satisfies the necessary condition

$$\mathcal{K}(\hat{q}, \hat{p}, t) = \mathcal{H}\left(Q(\hat{q}, \hat{p}, t), P(\hat{q}, \hat{p}, t), t\right) + \frac{\partial \mathcal{W}}{\partial t}$$

where $\partial \mathcal{W}/\partial t$ is called the remainder function.[Wintner, 1941]
ERTBP - Nechvile Transformation

The Nechvile transformation introduces pulsating coordinates through the generating function

\[ \mathcal{W}(q, \hat{p}, t) = \frac{r}{R} \hat{p}_r + \phi \hat{p}_\phi + \theta \hat{p}_\theta + \frac{\dot{R}}{2R} r^2 \]

where \( \dot{R} = (\partial R/\partial \nu) \dot{\nu} = R^2 e \sin \nu / p \). The corresponding state transformation equations are

\[
\begin{align*}
\hat{r} &= \frac{\partial \mathcal{W}}{\partial \hat{p}_r} = \frac{r}{R} \hat{p}_r \\
\hat{p}_r &= \frac{\partial \mathcal{W}}{\partial r} = \hat{p}_r + \dot{R} \hat{r} \\
\hat{\phi} &= \frac{\partial \mathcal{W}}{\partial \hat{p}_\phi} = \phi \hat{p}_\phi \\
p_\phi &= \frac{\partial \mathcal{W}}{\partial \phi} = \hat{p}_\phi \\
\hat{\theta} &= \frac{\partial \mathcal{W}}{\partial \hat{p}_\theta} = \theta \hat{p}_\theta \\
p_\theta &= \frac{\partial \mathcal{W}}{\partial \theta} = \hat{p}_\theta
\end{align*}
\]

It further executes a change of independent variable from time to true anomaly, \( t \rightarrow \nu \), and a scaling by \( \sqrt{p} \).
ERTBP - Normalized Hamiltonian Function

\[ \mathcal{H}(q, p, \nu) = \frac{1}{2} \left( r^2 + p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi} \right) - p_\theta \]

\[ - \frac{R}{p} \left( \frac{r^2}{2} + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} \right) \]

\[ r_1^2 = r^2 + \mu^2 - 2\mu r \sin \phi \cos \theta \]

\[ r_2^2 = r^2 + (1 - \mu)^2 + 2 (1 - \mu) r \sin \phi \cos \theta \]

The Hamiltonian equations of motion become

\[ \frac{dq_i}{d\nu} = \frac{\partial \mathcal{H}}{\partial p_i} \]

\[ \frac{dp_i}{d\nu} = - \frac{\partial \mathcal{H}}{\partial q_i} \]

\[ \frac{d\mathcal{H}}{d\nu} = \frac{\partial \mathcal{H}}{\partial \nu} \]
ERTBP - Normalized Hamiltonian Function

\[ \mathcal{H}(q, p, \nu) = \frac{1}{2} \left( r^2 + p_r^2 + \frac{p_{\phi}^2}{r^2} + \frac{p_{\theta}^2}{r^2 \sin^2 \phi} \right) - p_\theta \]

\[ - \frac{1}{1 + e \cos \nu} \left( \frac{r^2}{2} + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} \right) \]

\[ r_1^2 = r^2 + \mu^2 - 2\mu r \sin \phi \cos \theta \]

\[ r_2^2 = r^2 + (1 - \mu)^2 + 2(1 - \mu) r \sin \phi \cos \theta \]

The Hamiltonian equations of motion become

\[ \frac{dq_i}{d\nu} = \frac{\partial \mathcal{H}}{\partial p_i} \quad \frac{dp_i}{d\nu} = -\frac{\partial \mathcal{H}}{\partial q_i} \quad \frac{d\mathcal{H}}{d\nu} = \frac{\partial \mathcal{H}}{\partial \nu} \]
ERTBP - Lagrange Points

The ERTBP possesses five equilibrium points co-located with the five classic Lagrange points of the CRTBP.
ERTBP - Expansions

Expansion about the circular case:

\[
\frac{R}{p} = \sum_{n=0}^{\infty} (-e \cos \nu)^n
\]

Expansion about a triangular Lagrange point:

\[
\mathcal{H}(\delta q, \delta p, \nu) = \frac{1}{2} \left[ \left( \frac{1}{r_e} \delta p_\theta - 2\delta r \right)^2 + \delta p_r^2 + r_e^2 \delta \phi^2 + \frac{1}{r_e^2} \delta p_\phi^2 \right] \\
- \frac{3}{2} \sum_{n=0}^{\infty} (-e \cos \nu)^n \left[ r_e^2 \delta r^2 + \mu (1 - \mu) (\cos \theta_e \delta r - r_e \sin \theta_e \delta \theta)^2 \right] \\
+ O \left( \| (\delta q, \delta p) \|^3 \right)
\]

where \( r_e \cos \theta_e = \mu - \frac{1}{2} \) and \( r_e \sin \theta_e = \pm \sqrt{3/4} \).
The simplest form of the ERTBP is the linearized CRTBP. In this case, the system exhibits ideal behavior of integrability and stability wherein motion about the linearized, circular Lagrange points are in the form of harmonic oscillators. However, being neutrally stable, Lyapunov’s indirect method is inconclusive such that one may not infer the nonlinear stability based on the linear stability.

Instead, one may analyze the nonlinear system directly by applying canonical perturbation theory in order to normalize the nonlinear ERTBP about the ideal case of the linearized CRTBP. The ERTBP then assumes an integrable form, up to the order truncation in the normalization.
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4. Conclusions
Perturbation Theory - Von Zeipel’s Method

Jacobi, Poincaré and Von Zeipel developed the classic perturbation theory wherein the transformation generating function is expressed as a near-identity expansion series

\[ \mathcal{W}(q, \hat{p}, \nu) = q \cdot \hat{p} + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \mathcal{W}_n(q, \hat{p}, \nu) \]

to construct an ordered solution for the transformation

\[ \mathcal{H} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{H}_n(q, p, \nu) \rightarrow \mathcal{K} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{K}_n(\hat{q}, \hat{p}, \nu) \]

in the form

\[ \mathcal{K}_0(q, \hat{p}, \nu) = \mathcal{H}_0(q, \hat{p}, \nu) \]
\[ \mathcal{K}_1(q, \hat{p}, \nu) = \mathcal{H}_1(q, \hat{p}, \nu) + \frac{\partial \mathcal{W}_1}{\partial \nu} \]
\[ \mathcal{K}_2(q, \hat{p}, \nu) = \mathcal{H}_2(q, \hat{p}, \nu) + \frac{\partial \mathcal{W}_2}{\partial \nu} + \frac{\partial \mathcal{W}_1}{\partial q} \cdot \frac{\partial \mathcal{H}_1}{\partial p}(q, \hat{p}, \nu) \text{ etc.} \]
There are two major disadvantages to Von Zeipel’s method: it is not canonically invariant and the form of the generating function $\mathcal{W}(q, \hat{p}, \nu)$ is mixed requiring inversions.

In the 1960s, Deprit and Hori each independently derived an alternative method that constructs the transformation in terms of the Lie transform [Deprit,1969 & Hori,1966]

$$\mathcal{L}_{\mathcal{W}} \mathcal{H} \triangleq [\mathcal{H}, \mathcal{W}] = \sum_{i=1}^{N} \left( \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial \mathcal{W}}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial \mathcal{W}}{\partial q_i} \right)$$

In 1985, Varadi extended the method to Hamiltonian systems expanded about two parameters, but limited the analysis to autonomous systems and diffeomorphisms.[Varadi, 1985]

The two-parameter method is extended to non-autonomous systems and derived in the spirit of Deprit’s original formulation.
Canonical Perturbation Theory

- Deprit-Hori Lie Transform Method for Two-Parameter Non-Autonomous Systems
As an extension to the single parameter method, consider a doubly-expanded Hamiltonian system

\[ \mathcal{H}(q, p, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \mathcal{H}_{m,n}^{(0,0)}(q, p, \nu) \]

where \( \mathcal{H}_{0,0}^{(0,0)} \) is the unperturbed system. The Lie transform operator is applied to formulate the canonical transformation, but now through a pair of generating functions

\[ \mathcal{W}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \mathcal{W}_{m+1,n+1}(\hat{q}, \hat{p}, \nu) \]

\[ \mathcal{V}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \mathcal{V}_{m+1,n+1}(\hat{q}, \hat{p}, \nu) \]
The transformed Hamiltonian is expressed in expanded form

\[
K = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\epsilon^n \gamma^m}{n! m!} \left( H_{0,0}^{(m,n)} + R_{0,0}^{(m,n)} \right)
\]

where \( K_{0,0} = H_{0,0}^{(0,0)} \) under the identity transformation, \( H_{0,0}^{(m,n)} \) represents \( H(Q(\hat{q}, \hat{p}, \nu), P(\hat{q}, \hat{p}, \nu), \nu) \) and \( R_{0,0}^{(m,n)} \) represents the remainder function.

The transformed Hamiltonian is constructed from the recursive equations

\[
H_{m,n}^{(r,s)} = H_{m,n+1}^{(r,s-1)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} L_{W_{j+1,i+1}} H_{m-j,n-i}^{(r,s-1)}
\]

\[
H_{m,n}^{(r,s)} = H_{m+1,n}^{(r-1,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} L_{W_{j+1,i+1}} H_{m-j,n-i}^{(r-1,s)}
\]
DH Method - Deprit’s Pyramid

\[ \mathcal{H}^{(0,0)} \]

\[ \mathcal{H}^{(0,0)} \leftarrow \mathcal{H}^{(1,0)} \]

\[ \mathcal{H}^{(0,0)} \leftarrow \mathcal{H}_{0,0} \]

\[ \mathcal{H}^{(1,0)} \leftarrow \mathcal{H}_{1,0} \]

\[ \mathcal{H}^{(1,0)} \leftarrow \mathcal{H}^{(1,0)} \]

\[ \mathcal{H}^{(2,0)} \leftarrow \mathcal{H}^{(1,0)} \]

\[ \mathcal{H}^{(2,0)} \leftarrow \mathcal{H}_{0,0} \]

\[ \mathcal{H}^{(1,1)} \leftarrow \mathcal{H}_{1,1} \]

\[ \mathcal{H}^{(0,0)} \leftarrow \mathcal{H}_{1,1} \]

\[ \mathcal{H}^{(0,0)} \leftarrow \mathcal{H}_{0,1} \]

\[ \mathcal{H}^{(1,1)} \leftarrow \mathcal{H}_{0,1} \]

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\[ \mathcal{H}^{(0,1)} \leftarrow \mathcal{H}_{0,0} \]

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\[ \mathcal{H}^{(0,1)} \leftarrow \mathcal{H}_{1,0} \]

\[ \mathcal{H}^{(1,1)} \leftarrow \mathcal{H}_{1,0} \]

\[ \mathcal{H}^{(1,1)} \leftarrow \mathcal{H}_{0,0} \]
DH Method - Transformed Hamiltonian Function

and for the remainder function

\[ \mathcal{R}_{0,0}^{(m,n)} = \begin{cases} 
S_{0,0}^{(0,n-1)} & m = 0, n \neq 0 \\
T_{0,0}^{(m-1,0)} & m \neq 0, n = 0 \\
S_{0,0}^{(m,n-1)} = T_{0,0}^{(m-1,n)} & m, n \neq 0 
\end{cases} \]

where

\[ S_{m,n}^{(0,0)} = -\frac{\partial \mathcal{H}_{m+1,n+1}}{\partial \nu} \]

\[ T_{m,n}^{(0,0)} = -\frac{\partial \mathcal{V}_{m+1,n+1}}{\partial \nu} \]

\[ S_{m,n}^{(r,s)} = S_{m,n+1}^{(r,s-1)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} S_{m-j,n-i}^{(r,s-1)} \quad \text{(same for } T) \]

\[ S_{m,n}^{(r,s)} = S_{m+1,n}^{(r-1,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} S_{m-j,n-i}^{(r-1,s)} \quad \text{(same for } T) \]
DH Method - Ordered Differential Equations

The recursive algorithm generates a series of homological differential equations for each order of $\gamma$ and $\epsilon$

\[
\frac{\partial \mathcal{W}_{m,n}}{\partial \nu} - \mathcal{L} \mathcal{W}_{m,n} \mathcal{H}_{0,0} = Q_{m,n} - \mathcal{K}_{m,n} \quad \text{or} \quad \frac{\partial \mathcal{V}_{m,n}}{\partial \nu} - \mathcal{L} \mathcal{V}_{m,n} \mathcal{H}_{0,0} = P_{m,n} - \mathcal{K}_{m,n}
\]

including

\[
\mathcal{K}_{0,0} = \mathcal{H}_{0,0}^{(0,0)}
\]

\[
\mathcal{K}_{0,1} = -\frac{\partial \mathcal{W}_{1,1}}{\partial \nu} + \mathcal{L} \mathcal{W}_{1,1} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{0,1}^{(0,0)}
\]

\[
\mathcal{K}_{1,0} = -\frac{\partial \mathcal{V}_{1,1}}{\partial \nu} + \mathcal{L} \mathcal{V}_{1,1} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{1,0}^{(0,0)}
\]

\[
\mathcal{K}_{1,1} = -\frac{\partial \mathcal{W}_{2,1}}{\partial \nu} + \mathcal{L} \mathcal{W}_{2,1} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{1,1}^{(0,0)} + \ldots
\]

\[
= -\frac{\partial \mathcal{V}_{1,2}}{\partial \nu} + \mathcal{L} \mathcal{V}_{1,2} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{1,1}^{(0,0)} + \ldots
\]
Terms appearing in the expanded transformation containing “mixed parameters” (i.e. of the form $\epsilon^n \gamma^m$ with both $m$ and $n$ non-zero) may be obtained equivalently using either of the two generating functions $\mathcal{W}$ or $\mathcal{V}$ according to the Deprit commutation condition,

$$\frac{\partial \mathcal{V}}{\partial \epsilon} - \frac{\partial \mathcal{W}}{\partial \gamma} + \mathcal{L}_\mathcal{W} \mathcal{V} = 0$$

which term by term implies

$$\mathcal{V}_{m+1,n+2} - \mathcal{W}_{m+2,n+1} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{V}_{m-j+1,n-i+1} = 0$$

For example,

$$\mathcal{V}_{1,2} - \mathcal{W}_{2,1} + \mathcal{L}_{\mathcal{W}_{1,1}} \mathcal{V}_{1,1} = 0$$
DH Method - State Transformation Equations

The explicit state transformation equations are generated from

\[
q = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\epsilon^n \gamma^m}{n! m!} q^{(m,n)}_{0,0} (\hat{q}, \hat{p}, \nu)
\]

\[
p = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\epsilon^n \gamma^m}{n! m!} p^{(m,n)}_{0,0} (\hat{q}, \hat{p}, \nu)
\]

where

\[
q^{(0,0)}_{m,n} = \begin{cases} 
\hat{q} & m = n = 0 \\
0 & m + n > 0 
\end{cases} \quad \text{and} \quad p^{(0,0)}_{m,n} = \begin{cases} 
\hat{p} & m = n = 0 \\
0 & m + n > 0
\end{cases}
\]

\[
q^{(r,s+1)}_{m,n} = q^{(r,s)}_{m,n+1} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} L_{W_{j+1,i+1}} q^{(r,s)}_{m-j,n-i} \quad \text{(same for p)}
\]

\[
q^{(r+1,s)}_{m,n} = q^{(r,s)}_{m+1,n} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} L_{V_{j+1,i+1}} q^{(r,s)}_{m-j,n-i} \quad \text{(same for p)}
\]
The inverse transformation is constructed in terms of the inverse generating functions

\[
\hat{\mathcal{W}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \hat{\mathcal{W}}_{0,0}^{(m,n)}(q, p, \nu)
\]

\[
\hat{V} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \hat{\mathcal{V}}_{0,0}^{(m,n)}(q, p, \nu)
\]

derived from \(\hat{W}_{m,n}^{(0,0)} = -\mathcal{W}_{m,n}\) and \(\hat{V}_{m,n}^{(0,0)} = -\mathcal{V}_{m,n}\) and

\[
\hat{\mathcal{W}}_{m,n}^{(r,s+1)} = \hat{\mathcal{W}}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L} \hat{\mathcal{W}}_{j+1,i+1} \hat{\mathcal{W}}_{m-j,n-i}^{(r,s)}
\]

\[
\hat{\mathcal{W}}_{m,n}^{(r+1,s)} = \hat{\mathcal{W}}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L} \hat{\mathcal{V}}_{j+1,i+1} \hat{\mathcal{W}}_{m-j,n-i}^{(r,s)}
\]

(same for \(\hat{V}\))
DH Method - Inverse State Transformation Equations

The explicit inverse state transformation equations

\[
\hat{q} = \hat{Q}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \hat{q}_{0,0}^{(m,n)}(q, p, \nu)
\]

\[
\hat{p} = \hat{P}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \hat{p}_{0,0}^{(m,n)}(q, p, \nu)
\]

and generated in terms of the inverse generating functions

\[
\hat{q}_{m,n}^{(0,0)} = \begin{cases} 
q & m = n = 0 \\
0 & m + n > 0
\end{cases} \quad \text{and} \quad \hat{p}_{m,n}^{(0,0)} = \begin{cases} 
p & m = n = 0 \\
0 & m + n > 0
\end{cases}.
\]

\[
\hat{q}_{m,n}^{(r,s+1)} = \hat{q}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} L_{\hat{W}_{j+1,i+1}} \hat{q}_{m-j,n-i}^{(r,s)} \quad \text{(same for \( \hat{p} \))}
\]

\[
\hat{q}_{m,n}^{(r+1,s)} = \hat{q}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} L_{\hat{V}_{j+1,i+1}} \hat{q}_{m-j,n-i}^{(r,s)} \quad \text{(same for \( \hat{p} \))}
\]
DH Method - Recapitulation

- Expand the Hamiltonian function about two parameters: $\epsilon$ and $\gamma$.
- Generate the ordered homological equations in terms of $\mathcal{W}$ and $\mathcal{V}$.
- Prescribe a desired form for the transformed Hamiltonian $\mathcal{K}$.
- Solve the homological equations for the corresponding generating functions $\mathcal{W}$ and $\mathcal{V}$ term by term and using the Deprit commutation condition.
- Derive the forward state transformation equations.
- Derive the inverse generating functions $\hat{\mathcal{W}}$ and $\hat{\mathcal{V}}$.
- Derive the inverse state transformation equations in terms of $\hat{\mathcal{W}}$ and $\hat{\mathcal{V}}$. 
DH Method - Sketch of proof

- Append $\nu$ and $-\mathcal{H}$ as additional state variables.
- Define Deprit operators as
  \[ D_W \triangleq \frac{\partial}{\partial \epsilon} + \mathcal{L}_W \quad D_V \triangleq \frac{\partial}{\partial \gamma} + \mathcal{L}_V \]
  for which
  \[ \frac{\partial^n}{\partial \epsilon^n} \frac{\partial^m}{\partial \gamma^m} \left( F \bigg|_{(q,p)\to(\hat{q},\hat{p})} \right) = (D^n_W D^m_V F) \bigg|_{(q,p)\to(\hat{q},\hat{p})} \]

- The transformed Hamiltonian function is given roughly by
  \[ \hat{\mathcal{H}} \to D^n_W D^m_V \mathcal{H} \]

- Remainder function is generated similarly from $\frac{\partial W}{\partial \nu}, \frac{\partial V}{\partial \nu}$.
- State transformation equations are also generated similarly from their explicit expansions.
Canonical Perturbation Theory

- DH Transformation of the Elliptic Restricted Three-Body Problem
DH Transformation - Expanded Hamiltonian Function

Recall that the Hamiltonian function is doubly-expanded about the circular case and triangular Lagrange point as represented by

\[
\mathcal{H}(\delta q, \delta p, e, \gamma, \nu) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\gamma^m e^n}{m! n!} \mathcal{H}^{(0,0)}_{m,n}(\delta q, \delta p, \nu)
\]

where \(\gamma\) parameterizes the magnitude of the nonlinear terms and \(\mathcal{H}^{(0,0)}_{m,n}\) corresponds to terms of order \(\|(\delta q, \delta p)\|^{m+2}\) and \(e^n\).

The unperturbed system is the linearized CRTBP:

\[
\mathcal{H}^{(0,0)}_{0,0} = \frac{1}{2} \left[ \left( \frac{1}{r_e} \delta p_\theta - 2 \delta r \right)^2 + \delta p_r^2 + r_e^2 \delta \phi^2 + \frac{1}{r_e^2} \delta p_\phi^2 \right] - \frac{3}{2} \left[ r_e^2 \delta r^2 + \mu(1 - \mu) (\cos \theta_e \delta r - r_e \sin \theta_e \delta \theta)^2 \right]
\]
The linearized CRTBP exhibits stable motion in the form of harmonic oscillation about the linearized circular Lagrange point. The out-of-plane dynamics are in the form of a center as characterized by an eigenvalue of $\lambda = \pm i$ while the eigenvalues of the planar dynamics are given by the characteristic equation

$$
\lambda^4 + \lambda^2 + \frac{27}{4} \mu (1 - \mu) = 0
$$

which possesses four complex roots and a bifurcation at Routh’s critical mass ratio

$$
\mu_c = \left(1 - \frac{\sqrt{69}}{9}\right)/2 \approx 0.0385
$$
ERTBP - Motion about the Circular Lagrange Points

The bifurcation proceeds as follows:

\[ \mu < \mu_c \quad \rightarrow \quad \lambda = \pm i\omega_s, \pm i\omega_\ell \quad \text{(Neutrally Stable)} \]
\[ \mu = \mu_c \quad \rightarrow \quad \lambda^{(2)} = \pm i\sqrt{2}/2 \quad \text{(Unstable)} \]
\[ \mu > \mu_c \quad \rightarrow \quad \lambda = \pm a \pm ib \quad \text{(Unstable)} \]
DH Transformation - Unperturbed System

As such, the unperturbed system $H_{0,0}^{(0,0)}$ is linearly stable within the range $0 \leq \mu < \mu_c$. The ensuing harmonic oscillation about the Lagrange point is characterized by the natural frequencies

$$
\omega_s = \sqrt{1 + \frac{1 - 27\mu(1 - \mu)}{2}} \\
\omega_\ell = \sqrt{1 - \frac{1 - 27\mu(1 - \mu)}{2}}
$$

satisfying $0 < \omega_\ell < \frac{\sqrt{2}}{2} < \omega_s < 1$ and with $\omega_z = 1$ in the OOP.

- $\omega_\ell \rightarrow$ slow oscillatory mode with period $2\pi/\omega_\ell$
- $\omega_s \rightarrow$ fast, epicyclic mode with period $2\pi/\omega_s$

Applying the Breakwell-Pringle linear transformation yields

$$
H_{0,0}^{(0,0)} = \frac{1}{2} (\omega_s^2 q_s^2 + p_s^2) - \frac{1}{2} (\omega_\ell^2 q_\ell^2 + p_\ell^2) + \frac{1}{2} (\omega_z^2 q_z^2 + p_z^2)
$$

in terms of generalized short- and long-period (SLP) variables $(q_s, p_s, q_\ell, p_\ell, q_z, p_z)$.
DH Transformation - Integrable Unperturbed System

Introducing action-angle variables \((I, \theta)\)

\[
I_i \triangleq \frac{\omega_i^2 q_i^2 + p_i^2}{2\omega_i} \quad \tan \theta_i \triangleq \frac{\omega_i q_i}{p_i}
\]

The unperturbed system takes the integrable form

**Unperturbed System**

\[
\mathcal{H}^{(0,0)}_0 = \omega_s I_s - \omega_\ell I_\ell + \omega_z I_z
\]

where the angular variables are cyclic (ignorable)

\[
\frac{d\theta_i}{d\nu} = \frac{\partial \mathcal{H}^{(0,0)}_0}{\partial I_i} = \pm \omega_i
\]

\[
\frac{dI_i}{d\nu} = -\frac{\partial \mathcal{H}^{(0,0)}_0}{\partial \theta_i} = 0
\]

for \(i = s, \ell, \text{and } z\).
when we have a system of $n$ (independent) first integrals in involution, the Hamiltonian system is completely integrable . . . if the $n$ first integrals are in involution, then the corresponding momenta are also constant and we have all the possible $2n$ independent first integrals [Boccaletti, 1999]

A system is integrable if its phase space is “foliated” by invariant submanifolds parameterized the integrals of motion.
The unperturbed system $\mathcal{H}_{0,0}^{(0,0)}$ is autonomous and integrable with natural frequencies $\omega_s$, $\omega_\ell$ and $\omega_z = 1$ while the linearly normalized perturbed system

$$\mathcal{H}(\theta, I, \nu) = \omega_s I_s - \omega_\ell I_\ell + \omega_z I_z + \sum_{m+n>0} \frac{\gamma^m e^n}{m! n!} \mathcal{H}_{m,n}^{(0,0)}(\theta, I, \nu)$$

is non-autonomous and non-integrable.

The true anomaly and angular variables only appear in the form $\mathcal{H}_{m,n}^{(0,0)} = \mathcal{H}_{m,n}^{(0,0)}(I_i, \cos \theta_i, \sin \theta_i, \cos \nu, \sin \nu)$ for $m + n > 0$.

Therefore, the DH method may be applied to average out these periodic terms into a locally integrable form reminiscent of the unperturbed case.
Based on the form of $\mathcal{H}_{0,0}^{(0,0)}$, the homological equation is

$$\left( \frac{\partial}{\partial \nu} + \omega_s \frac{\partial}{\partial \theta_s} - \omega_\ell \frac{\partial}{\partial \theta_\ell} + \omega_z \frac{\partial}{\partial \theta_z} \right) W_{i+1,j} = Q_{i,j} - K_{i,j}$$

for which the transformed Hamiltonian function is defined by the periodic average

$$K_{i,j} \triangleq <Q_{i,j}> = \frac{1}{4(2\pi)^4} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} Q_{i,j} \ d\nu \ d\theta_s \ d\theta_\ell \ d\theta_z$$

such that $K_{i,j} = K_{i,j}(\hat{\mathcal{I}})$ under the transformation $(\theta, \mathcal{I}) \rightarrow (\hat{\theta}, \hat{\mathcal{I}})$. 
The DH Transformation - Transformed System

The ERTBP Hamiltonian function is transformed into the integrable form up to the order of truncation

\[
\mathcal{K} = \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) \hat{I}_s - \left( \omega_\ell + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{\ell,n} \right) \hat{I}_\ell + \omega_z \hat{I}_z \\
+ \left( \alpha_{ss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss,n} \right) \hat{I}_s^2 + \left( \alpha_{ll} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ll,n} \right) \hat{I}_\ell^2 \\
+ \left( \alpha_{zz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{zz,n} \right) \hat{I}_z^2 + \left( \alpha_{sl} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sl,n} \right) \hat{I}_s \hat{I}_\ell \\
+ \left( \alpha_{sz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sz,n} \right) \hat{I}_s \hat{I}_z + \left( \alpha_{lz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{lz,n} \right) \hat{I}_\ell \hat{I}_z \\
+ O(\hat{I}_i^3)
\]
The coefficients of the transformed Hamiltonian function are:

\[
\tilde{\omega}_{s,1} = \frac{\omega_s(1 - \omega_s^2)(7 - 6\omega_s^2)}{2(1 - 2\omega_s^2)(1 - 4\omega_s^2)}
\]

\[
\alpha_{ss} = \frac{\omega_s^2(81 - 696\omega_s^2 + 124\omega_s^4)}{72(1 - 2\omega_s^2)^2(1 - 5\omega_s^2)}
\]

\[
\alpha_{\ell\ell} = \frac{\omega_s^2(81 - 696\omega_\ell^2 + 124\omega_\ell^4)}{72(1 - 2\omega_\ell^2)^2(1 - 5\omega_\ell^2)}
\]

\[
\alpha_{zz} = -\frac{2\omega_s^2\omega_\ell^2}{3(12 + \omega_s^2\omega_\ell^2)}
\]

\[
\alpha_{s\ell} = -\frac{\omega_s\omega_\ell(43 + 64\omega_s^2\omega_\ell^2)}{3(1 - 2\omega_s^2)(1 - 2\omega_\ell^2)(1 - 5\omega_s^2)(1 - 5\omega_\ell^2)}
\]

\[
\tilde{\omega}_{\ell,1} = \frac{\omega_\ell(1 - \omega_\ell^2)(7 - 6\omega_\ell^2)}{2(1 - 2\omega_\ell^2)(1 - 4\omega_\ell^2)}
\]

\[
\alpha_{sz} = -\frac{16\omega_s\omega_\ell^2}{3(4 - 9\omega_s^2 + 2\omega_s^4)}
\]

\[
\alpha_{\ell z} = \frac{16\omega_s^2\omega_\ell}{3(4 - 9\omega_s^2 + 2\omega_\ell^4)}
\]
DH Transformation - Transformed System

The transformed equations of motion and solutions are

\[
\frac{d\hat{\theta}_s}{d\nu} = \frac{\partial K}{\partial \hat{I}_s} = \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) \\
+ 2 \left( \alpha_{ss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss,n} \right) \hat{I}_s + \left( \alpha_{sl} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sl,n} \right) \hat{I}_l \\
+ \left( \alpha_{sz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sz,n} \right) \hat{I}_z + O \left( \hat{I}_i^2 \right) = \Omega_s
\]

\[
\frac{d\hat{I}_s}{d\nu} = -\frac{\partial K}{\partial \hat{\theta}_s} = 0
\]

\[
\hat{\theta}_s(\nu) = \hat{\theta}_{s,0} \pm \Omega_s (\nu - \nu_0) \pmod{2\pi}
\]

\[
\hat{I}_s(\nu) = \hat{I}_{s,0}
\]

(likewise for \( \ell \) and \( z \))
Outline

1. Elliptic Restricted Three-Body Problem
   - Spacecraft Dynamics
   - Hamiltonian Formulation

2. Canonical Perturbation Theory
   - Deprit-Hori Lie Transform Method for Two-Parameter Non-Autonomous Systems
   - DH Transformation of the ERTBP

3. Dynamical Analysis
   - Stability Analysis
   - Local Integrals of Motion
   - Control

4. Conclusions
Dynamical Analysis

- Stability Analysis
Stability Analysis - Linear, Non-Circular Stability

The linearized system in the transformed phase space is

\[ K_{0,i} = \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) \hat{I}_s - \left( \omega_\ell + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{\ell,n} \right) \hat{I}_\ell + \omega_z \hat{I}_z \]

The linear stability is characterized by the Floquet exponents

\[ \sigma = \pm i \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) \quad \text{and} \quad \lambda = \mp i \left( \omega_\ell + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{\ell,n} \right) \]

for which the Floquet multipliers \( \exp(2\pi\sigma) \) and \( \exp(2\pi\lambda) \) must have non-positive real parts to achieve linear stability.
Stability Analysis - Linear, Non-Circular Stability

Danby’s Stability Curve:

Figure: Danby’s Stability Curve
Stability Analysis - Nonlinear, Non-Circular Stability

KAM theorem: For a “nearly-integrable” Hamiltonian system

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_n \]

for a sufficiently small Hamiltonian perturbation most non-resonant invariant tori do not vanish but are only slightly deformed, so that in the phase space of the perturbed system \([\mathcal{H}_0 + \mathcal{H}_n]\) there are invariant tori densely filled with [quasi-periodic] phase curves winding around them, [Boccaletti, 1999]

Complications:

- KAM theorem only implies stability for 2-dimensional systems (due to Arnold diffusion)
- Requires non-degeneracy in the unperturbed system, but \(\mathcal{H}_0 = \vec{\omega} \cdot \vec{I}\) is degenerate
Arnold theorem: Consider the 2-D Hamiltonian system

\[ \mathcal{K} = \mathcal{K}_2 + \mathcal{K}_4 + \ldots + \mathcal{K}_{2N} + \mathcal{K}_{2N+1} \]

where

- \( \mathcal{K} \) is real-analytic in the neighborhood of the origin
- \( \mathcal{K}_2 \) is in the form \( \omega_s \hat{I}_s - \omega_\ell \hat{I}_\ell \)
- \( \mathcal{K}_{2k} \) is a homogeneous polynomial of degree \( k \) in \( \hat{I}_s \) and \( \hat{I}_\ell \)
- \( \mathcal{K}_{2N+1} \) has a series expansion of degree \( \geq 2N + 1 \)
- The natural frequencies \( \omega_i \) are non-resonant

The origin of the system is stable if for at least one \( k \) in the range \( 1 \leq k \leq N \), the system satisfies the Arnold condition

\[ D_{2k} = \mathcal{K}_{2k}(\hat{I}_s = \omega_\ell, \hat{I}_\ell = \omega_s) \neq 0 \]
The fourth- and sixth-order Arnold condition in the transformed phase space is

\[
D_4 = \left( \alpha_{ss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss,n} \right) \omega^2_{\ell} + \left( \alpha_{ll} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ll,n} \right) \omega^2_{s} \\
+ \left( \alpha_{sl} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sl,n} \right) \omega_{s} \omega_{\ell} \neq 0
\]

\[
D_6 = \left( \alpha_{sss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sss,n} \right) \omega^3_{\ell} + \left( \alpha_{lll} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{lll,n} \right) \omega^3_{s} \\
+ \left( \alpha_{ssl} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ssl,n} \right) \omega_{s} \omega^2_{\ell} + \left( \alpha_{ssl} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ssl,n} \right) \omega^2_{s} \omega_{\ell} \neq 0
\]
Stability Analysis - Nonlinear, Circular Stability

In the CRTBP system, the fourth-order Arnold condition reduces to

\[
D_4 = \alpha_{ss}\omega_s^2 + \alpha_{\ell\ell}\omega_s^2 + \alpha_{s\ell}\omega_s\omega_{\ell} = \frac{\omega^4_\ell(81 - 696\omega_s^2 + 124\omega_s^4)}{72(1 - 2\omega_s^2)^2(1 - 5\omega_s^2)} + \frac{\omega^4_s(81 - 696\omega_\ell^2 + 124\omega_\ell^4)}{72(1 - 2\omega_\ell^2)^2(1 - 5\omega_\ell^2)} - \frac{\omega_s^2\omega_\ell^2(43 + 64\omega_s^2\omega_\ell^2)}{3(1 - 2\omega_s^2)(1 - 2\omega_\ell^2)(1 - 5\omega_s^2)(1 - 5\omega_\ell^2)}
\]

\[
= -\frac{36 - 541\omega_s^2\omega_\ell^2 + 644\omega_s^4\omega_\ell^4}{8(1 - 4\omega_s^2\omega_\ell^2)(4 - 25\omega_s^2\omega_\ell^2)} \neq 0
\]

which is satisfied for all mass ratios except

\[
\mu_4 = \frac{1449 - \sqrt{483(3265 + 2\sqrt{199945})}}{2898} \approx 0.0109137
\]
However, the sixth-order Arnold condition is

\[ D_6 = \left( -16096320 + 578209968\omega^2_{\ell}\omega^2_s - 5879019660\omega^4_{\ell}\omega^4_s \right. \]

\[ + 23361243081\omega^6_{\ell}\omega^6_s - 32843706320\omega^8_{\ell}\omega^8_s - 104264873152\omega^{10}_{\ell}\omega^{10}_s \]

\[ + 481275622400\omega^{12}_{\ell}\omega^{12}_s + 94280800000\omega^{14}_{\ell}\omega^{14}_s \right) \]

\[ / \left( 20736\omega_{\ell}\omega_s (\omega^2_{\ell} - \omega^2_s)^5 (-4 + 25\omega^2_{\ell}\omega^2_s)^3 (-9 + 100\omega^2_{\ell}\omega^2_s) \right) \]

\[ \neq 0 \]

which is satisfied at \( \mu = \mu_4 \) (\( D_6 \approx 66 \)). Thus, the circular triangular Lagrange points are nonlinearly stable for all non-resonant mass ratios within \( 0 < \mu < \mu_c \).
In the ERTBP system, Arnold’s stability conditions depend on infinite series in $e$. Up to order $e^6$, the numerically generated stability curves are shown below.
The only location at which the nonlinear stability remains in question is the point where the green and red curves intersect near $\mu = 0.0102$ and $e = 0.075$. However, the eighth-order Arnold condition

$$D_8 = \left( \alpha_{ssss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ssss,n} \right) \omega_4^4 + \left( \alpha_{llll} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{llll,n} \right) \omega_s^4$$

$$+ \left( \alpha_{ss\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss\ell,n} \right) \omega_s^3 \omega_\ell + \left( \alpha_{s\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell\ell,n} \right) \omega_s^2 \omega_\ell^2$$

$$+ \left( \alpha_{\ell\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell\ell\ell,n} \right) \omega_3^3 \omega_\ell \neq 0$$

is satisfied at the intersection point.

Thus, the elliptic triangular Lagrange points are nonlinearly stable for all non-resonant mass ratios within $0 < \mu < \mu_c$. 
Dynamical Analysis

- Local Integrals of Motion
Local Integrals of Motion

Recall the transformed planar equations of motion for the action-type variables are

\[ \frac{d\hat{I}_s}{d\nu} = -\frac{\partial K}{\partial \hat{\theta}_s} = 0 \]
\[ \frac{d\hat{I}_\ell}{d\nu} = -\frac{\partial K}{\partial \hat{\theta}_\ell} = 0 \]
\[ \frac{d\hat{I}_z}{d\nu} = -\frac{\partial K}{\partial \hat{\theta}_z} = 0 \]

such that the action-type variables are integrals of motion

\[ \hat{I}_i(\nu) = \hat{I}_i,0 \]

in the transformed phase space and up to the order of truncation in the DH transformation.
Local Integrals of Motion - Jacobi-Type Equation

Since the action-type variables represent local integrals of motion, one may generate local level sets that foliate the phase space in the vicinity of the Lagrange point.

The system momentum in the SLP phase space satisfies

\[ \hat{p}_s^2 + \hat{p}_\ell^2 + \hat{p}_z^2 = 2 \left( \omega_s \hat{I}_s + \omega_\ell \hat{I}_\ell + \omega_z \hat{I}_z \right) - \left( \omega_s^2 \hat{q}_s^2 + \omega_\ell^2 \hat{q}_\ell^2 + \omega_z^2 \hat{q}_z^2 \right) \]

where

\[ C = \omega_s \hat{I}_s + \omega_\ell \hat{I}_\ell + \omega_z \hat{I}_z \]

serves as an integral in the tradition of Jacobi’s integral.
Local Integrals of Motion

The local integral is plotted along the numerically generated solutions for the Earth-Moon system.

(a) 0\textsuperscript{th}-Order DH Solution
(b) 2\textsuperscript{nd}-Order DH Solution

Figure: Local Integral of Motion for the Earth-Moon System
Local Integrals of Motion

Figure: Local Level Sets and Trajectory
Local Integrals of Motion - Eccentric Earth-Moon

(a) $0^{th}$-Order DH Solution
(b) $2^{nd}$-Order DH Solution

Figure: Local Integral of Motion for the Eccentric Earth-Moon
Local Integrals of Motion - Eccentric Earth-Moon

**Figure:** Local Level Sets and Trajectory
Dynamical Analysis

- Control
In the framework of the DH method, control is applied through the addition of a non-autonomous forcing function $U(q, p, \epsilon, \gamma, \nu)$ to form the controlled Hamiltonian function

$$\mathcal{H}_c(q, p, \epsilon, \gamma, \nu) = \mathcal{H}(q, p, \epsilon, \gamma, \nu) + \mathcal{U}(q, p, \epsilon, \gamma, \nu)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! \ n!} \left( \mathcal{H}^{(0,0)}_{m,n}(q, p, \nu) + \mathcal{U}^{(0,0)}_{m,n}(q, p, \nu) \right)$$

yielding

$$\mathcal{K}_c = \mathcal{H}_c \left( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \right) + \mathcal{R}_c (\hat{q}, \hat{p}, \epsilon, \gamma, \nu)$$

$$= \mathcal{H} \left( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \right)$$

$$+ \mathcal{U} \left( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \right) + \mathcal{R}_c (\hat{q}, \hat{p}, \epsilon, \gamma, \nu)$$
The controlled homological equation is

\[
\frac{\partial \mathcal{W}_{c,i+1,j}}{\partial \nu} - \mathcal{L}_{\mathcal{W}_{c,i+1,j}} \mathcal{H}^{(0,0)}_{c,0,0} = Q_{c,i,j} - K_{c,i,j}
\]

where $Q_{c,i,j}$ is known \textit{a priori} and $K_{c,i,j}$ is the transformed controlled Hamiltonian.

Zero control is applied to the unperturbed system

\[
\mathcal{U}^{(0,0)}_{0,0} \equiv 0 \quad \implies \quad \mathcal{H}_{c,0,0}^{(0,0)} = \mathcal{H}^{(0,0)}_{0,0}
\]

and the controlled generating functions are de-coupled into a sum of independent generating functions

\[
\mathcal{W}_{c} = \mathcal{W} + \mathcal{W}_{u} \\
\mathcal{V}_{c} = \mathcal{V} + \mathcal{V}_{u}
\]
Control

The controlled homological equation then takes the form

\[ \mathcal{K}_{c,i,j} = - \frac{\partial \mathcal{W}_{i+1,j}}{\partial \nu} + \mathcal{L} \mathcal{W}_{i+1,j} \mathcal{H}_{0,0}^{(0,0)} + Q_{i,j} \]

\[ - \frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L} \mathcal{W}_{u,i+1,j} \mathcal{H}_{0,0}^{(0,0)} + Q_{u,i,j} \]

\[ + U_{i,j}^{(0,0)} + Q_{c,i,j} \]

where \( Q_{i,j}, Q_{u,i,j} \) and \( Q_{c,i,j} \) are known \textit{a priori} based on the original Hamiltonian function and terms of lesser order.
Control

The controlled homological equation then takes the form

\[ K_{c,i,j} = -\frac{\partial W_{i+1,j}}{\partial \nu} + L W_{i+1,j} H_{0,0}^{(0,0)} + Q_{i,j} = K_{i,j} \]

\[ -\frac{\partial W_{u,i+1,j}}{\partial \nu} + L W_{u,i+1,j} H_{0,0}^{(0,0)} + Q_{u,i,j} \]

\[ + U_{i,j}^{(0,0)} + Q_{c,i,j} \]

General Control Law

\[ K_{c,i,j} - K_{i,j} = -\frac{\partial W_{u,i+1,j}}{\partial \nu} + L W_{u,i+1,j} H_{0,0}^{(0,0)} + U_{i,j}^{(0,0)} \]

\[ + Q_{u,i,j} + Q_{c,i,j} \]

where \( U_{i,j}^{(0,0)} \) and \( W_{u,i+1,j} \) (likewise \( V_{u,i,j+1} \)) are the control functions designed to achieve a desired form for \( K_{c,i,j} \).
Secular Control

\( \mathcal{W}_u = \mathcal{V}_u = 0 \)

\( \mathcal{U}_{i,j}^{(0,0)} = \mathcal{K}_{c,i,j} - \mathcal{K}_{i,j} - \mathcal{Q}_{c,i,j} \)

\( \mathcal{U}_{0,1}^{(0,0)} = \mathcal{K}_{c,0,1} - \mathcal{K}_{0,1} \)

\( \mathcal{U}_{1,0}^{(0,0)} = \mathcal{K}_{c,1,0} - \mathcal{K}_{1,0} \)

\( \mathcal{U}_{0,2}^{(0,0)} = \mathcal{K}_{c,0,2} - \mathcal{K}_{0,2} - \mathcal{L}_w_{1,1} \left( \mathcal{U}_{0,1}^{(0,0)} + \mathcal{U}_{0,0}^{(0,1)} \right) \)

\( \mathcal{U}_{2,0}^{(0,0)} = \mathcal{K}_{c,2,0} - \mathcal{K}_{2,0} - \mathcal{L}_v_{1,1} \left( \mathcal{U}_{1,0}^{(0,0)} + \mathcal{U}_{0,0}^{(1,0)} \right) \)

\( \mathcal{U}_{1,1}^{(0,0)} = \mathcal{K}_{c,1,1} - \mathcal{K}_{1,1} - \mathcal{L}_v_{1,1} \mathcal{U}_{0,1}^{(0,0)} - \mathcal{L}_w_{1,1} \mathcal{U}_{0,0}^{(1,0)} \)

\( = \mathcal{K}_{c,1,1} - \mathcal{K}_{1,1} - \mathcal{L}_w_{1,1} \mathcal{U}_{1,0}^{(0,0)} - \mathcal{L}_v_{1,1} \mathcal{U}_{0,0}^{(0,1)} \)
Consider the control problem of stabilizing motion about an elliptic triangular Lagrange point of the Earth-Moon system where the controlled Hamiltonian function is prescribed by the corresponding circular form, that is,

\[ K_{c,i,j} = \begin{cases} 
  0 & j > 0 \\
  K_{i,j} & j = 0
\end{cases} \]

which effectively eliminates all the non-circular perturbations.

**Figure**: Circular Solution
Control - Secular Control

For the eccentric Earth-Moon system, $\mu = 0.0124$ and $e = 0.2$, the uncontrolled and secularly controlled solutions are

(a) Uncontrolled Trajectory  
(b) Controlled Trajectory
Control - Secular Control

(a) Uncontrolled $\delta q_x$

(b) Controlled $\delta q_x$

(c) Uncontrolled $\delta p_x$

(d) Controlled $\delta p_x$
Outline

1. Elliptic Restricted Three-Body Problem
   - Spacecraft Dynamics
   - Hamiltonian Formulation

2. Canonical Perturbation Theory
   - Deprit-Hori Lie Transform Method for Two-Parameter Non-Autonomous Systems
   - DH Transformation of the ERTBP

3. Dynamical Analysis
   - Stability Analysis
   - Local Integrals of Motion
   - Control

4. Conclusions
Recapitulation:

- The Hamiltonian function of the ERTBP is non-autonomous and non-integrable.
- Perturbation theory is applied to represent the ERTBP as a close expansion about the integrable linearized CRTBP.
- The system is normalized using the DH method extended to non-autonomous Hamiltonian functions expanded about two parameters.
- The transformed ERTBP dynamics are in the form of perturbed harmonic oscillators.
Conclusions - Recap

- The triangular elliptic Lagrange points are nonlinearly stable for all values of $\mu$ and $e$ under Danby’s curve excepting those that exhibit resonance in the natural frequencies.

- The local foliations in the phase space are represented by applying the action-type variables as local integrals of motion such that regions of stability are approximated within the pulsating synodic frame.

- The incorporation of control terms into the DH transformation results in an additional series of homological equations for the control terms and the corresponding control generating functions.

- Secular control is achieved for the non-circular ERTBP.
Conclusions - Original Contributions

Original Contributions:

- The extension of the DH method to non-autonomous, two-parameter Hamiltonian systems.
- Incorporation of control into the extended DH method.
- Application of the extended DH method to motion about the triangular elliptic Lagrange points.
- Nonlinear stability of the triangular elliptic Lagrange points.
- Generation of curves of zero momentum from local integrals of motion.
- Preliminary feedback stabilization about the elliptic Lagrange points using the DH method control laws.
Future Work:

- Arnold Diffusion in the ERTBP.
- Nonlinear and optimal control theory within DH method and applied to ERTBP.
- Motion beyond the local phase space.
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Thank You for Attending

Once again, I’d like to thank Dr. Chichka and all my committee members for their time and expertise in reviewing my dissertation and attending this defense.

Questions?
Back-up Slides
Outline

5 Elliptic Restricted Three-Body Problem

6 Perturbation Theory
   - HJE & Von Zeipel’s Method

7 DH Single Parameter Method
   - DH Method

8 Extended DH Method Proofs

9 DH Transformation of the ERTBP

10 DH Method Control

11 Nonlinear Oscillator with Damping
Outline

5. Elliptic Restricted Three-Body Problem

6. Perturbation Theory
   - HJE & Von Zeipel’s Method

7. DH Single Parameter Method
   - DH Method

8. Extended DH Method Proofs

9. DH Transformation of the ERTBP

10. DH Method Control

11. Nonlinear Oscillator with Damping
ERTBP - Equilibrium Points

The ERTBP equilibrium points are defined by the equations
\[ \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial q_i} = 0, \]
which yield
\[ \phi_e = \pi/2 \quad p_{r,e} = 0 \quad p_{\phi,e} = 0 \quad p_{\theta,e} = r_e^2 \]
and
\[ 0 = r_e - r_e \left( \frac{1 - \mu}{r_{1,e}^3} + \frac{\mu}{r_{2,e}^3} \right) + \mu(1 - \mu) \cos \theta_e \left( \frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3} \right) \]
\[ 0 = \mu(1 - \mu) r_e \sin \theta_e \left( \frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3} \right) \]

Since the equilibrium equations are independent of the true anomaly and eccentricity, they are fixed relative to the synodic, pulsating frame and co-located with the Lagrange points.
ERTBP - Lagrange Points

The Lagrange points in the ERTBP system are denoted as $L_1$, $L_2$, $L_3$, $L_4$, and $L_5$. These points are significant in dynamical systems and celestial mechanics, particularly in the study of gravitational dynamics in the presence of perturbations.
ERTBP - Lagrange Points

Collinear Lagrange points, $L_1$, $L_2$ and $L_3$:

$$0 = r_e - r_e \left( \frac{1 - \mu}{r_{1,e}^3} + \frac{\mu}{r_{2,e}^3} \right) \pm \mu(1 - \mu) \left( \frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3} \right)$$

where $\theta_e = 0$ and $r_{1,e} = |r_e \mp \mu|$ and $r_{2,e} = |r_e \pm (1 - \mu)|$.

Triangular Lagrange points, $L_4$ and $L_5$:

$$r_e \cos \theta_e = \mu - \frac{1}{2} \quad p_{r,e} = 0$$

$$\phi_e = \frac{\pi}{2} \quad p_{\phi,e} = 0$$

$$r_e \sin \theta_e = \pm \sqrt{3/4} \quad p_{\theta,e} = \left( \mu - \frac{1}{2} \right)^2 + \frac{3}{4}$$
ERTBP - Motion about the Circular Lagrange Points

The linearized out-of-plane dynamics are in the form of a center about the Lagrange point for both the ERTBP and CRTBP. For the planar dynamics, set $e = 0$ and linearize the circular Hamiltonian function about a triangular Lagrange point

$$\mathcal{H} = \frac{1}{2} \left[ \left( \frac{1}{r_e} \delta p_\theta - 2 \delta r \right)^2 + \delta p_r^2 + r_e^2 \delta \phi^2 + \frac{1}{r_e^2} \delta p_\phi^2 \right]$$

$$- \frac{3}{2} \left[ r_e^2 \delta r^2 + \mu (1 - \mu) (\cos \theta_e \delta r - r_e \sin \theta_e \delta \theta)^2 \right]$$

The characteristic equation of the linearized system is

$$\lambda^4 + \lambda^2 + \frac{27}{4} \mu (1 - \mu) = 0$$

which possesses four complex roots and a bifurcation at

$$\mu_c = \left( 1 - \frac{\sqrt{69}}{9} \right) / 2 \approx 0.0385$$
The bifurcation proceeds as follows:

\[ \mu < \mu_c \rightarrow \lambda = \pm i \omega_s, \pm i \omega_l \] (Stable)

\[ \mu = \mu_c \rightarrow \lambda^{(2)} = \pm i \sqrt{2}/2 \] (Unstable)

\[ \mu > \mu_c \rightarrow \lambda = \pm a \pm ib \] (Unstable)
ERTBP - Jacobi’s Integral

Representing the Hamiltonian function as

\[ \mathcal{H}(q, p, \nu) = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi} \right) - p_\theta + \frac{R}{p} U(r, \phi, \theta) \]

the corresponding 2nd-order equations of motion are

\[ \frac{d^2 r}{d\nu^2} - r \left( \frac{d\phi}{d\nu} \right)^2 - r \sin^2 \phi \left( 1 + \frac{d\theta}{d\nu} \right)^2 = -\frac{R}{p} \frac{\partial U}{\partial r} \]

\[ \frac{d^2 \phi}{d\nu^2} + \frac{2}{r} \frac{dr}{d\nu} \frac{d\phi}{d\nu} - \cos \phi \sin \phi \left( 1 + \frac{d\theta}{d\nu} \right)^2 = -\frac{1}{r^2} \frac{R}{p} \frac{\partial U}{\partial \phi} \]

\[ \frac{d^2 \theta}{d\nu^2} + \frac{2}{r} \frac{dr}{d\nu} \left( 1 + \frac{d\theta}{d\nu} \right) + \frac{2 \cos \phi}{\sin \phi} \frac{d\phi}{d\nu} \left( 1 + \frac{d\theta}{d\nu} \right) = -\frac{1}{r^2 \sin^2 \phi} \frac{R}{p} \frac{\partial U}{\partial \theta} \]

Multiplying each in turn by \( \frac{dr}{d\nu}, \frac{r^2 d\phi}{d\nu}, \) and \( r^2 \sin^2 \phi \frac{d\theta}{d\nu} \) and summing them together yields
ERTBP - Jacobi’s Integral

\[
\frac{1}{2} \frac{d}{d\nu} \left( p_r^2 + \frac{p_{\phi}^2}{r^2} + \frac{p_{\theta}^2}{r^2 \sin^2 \phi} \right) = -\frac{dp_{\theta}}{d\nu} + \frac{R}{p} \frac{dU}{d\nu}
\]

For the non-circular case, \( R/p = 1/(1 + e \cos \nu) \) such that the right-hand side is not a total derivative. However, for the circular case, \( R/p = 1 \), and the above equation integrates to

\[
p_r^2 + \frac{p_{\phi}^2}{r^2} + \frac{p_{\theta}^2}{r^2 \sin^2 \phi} = 2p_{\theta} - 2U + C
\]

where \( C \) is Jacobi’s constant and equal to twice the circular Hamiltonian function

\[
\mathcal{H} \bigg|_{e=0} = \frac{C}{2}
\]
ERTBP - Jacobi’s Integral

Figure: Hill Regions in the CRTBP
Outline

5 Elliptic Restricted Three-Body Problem

6 Perturbation Theory
   - HJE & Von Zeipel’s Method

7 DH Single Parameter Method
   - DH Method

8 Extended DH Method Proofs

9 DH Transformation of the ERTBP

10 DH Method Control

11 Nonlinear Oscillator with Damping
Jacobi postulated a canonical transformation \((q, p) \rightarrow (\hat{q}, \hat{p})\) of a Hamiltonian system \(\mathcal{H}(q, p, \nu)\) as generated from \(\mathcal{W}(q, \hat{p}, \nu)\) such that

\[
\begin{align*}
p_i &= \frac{\partial}{\partial q_i} \mathcal{W}(q, \hat{p}, \nu) \\
\hat{q}_i &= \frac{\partial}{\partial \hat{p}_i} \mathcal{W}(q, \hat{p}, \nu)
\end{align*}
\]

and the transformed Hamiltonian function is

\[
\mathcal{K}(\hat{q}, \hat{p}, \nu) = \mathcal{H}(q, \frac{\partial \mathcal{W}}{\partial q}, \nu) + \frac{\partial \mathcal{W}}{\partial \nu}
\]

where \(\frac{\partial \mathcal{W}}{\partial \nu}\) constitutes a remainder function. If \(\mathcal{K}\) is set to zero, implying complete integrability, the above equation is called the Hamilton-Jacobi equation.
Perturbation Theory - Expansions

Jacobi further postulated an expansion of the Hamiltonian function in the form of a perturbed integrable system

\[ \mathcal{H} = \mathcal{H}_0 + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \mathcal{H}_n \]

where \( \epsilon \) parameterizes the order of the perturbation terms and the unperturbed system \( \mathcal{H}_0 \) is autonomous and integrable.

The aim of perturbation theory is to derive a canonical transformation of the Hamiltonian system in the expanded form

\[ \mathcal{H} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{H}_n(q, p, \nu) \rightarrow \mathcal{K} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{K}_n(\hat{q}, \hat{p}, \nu) \]

such that the transformed Hamiltonian function \( \mathcal{K}(\hat{q}, \hat{p}, \nu) \) possesses ideal properties of integrability and autonomy.
Perturbation Theory - Von Zeipel’s Method

Von Zeipel’s method further introduces a near-identity generating function

\[ \mathcal{W}(q, \hat{p}, \nu) = q \cdot \hat{p} + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \mathcal{W}_n(q, \hat{p}, \nu) \]

such that the transformed Hamiltonian function is

\[ \mathcal{K}(q, \hat{p}, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{K}_n(q, \hat{p}, \nu) \]

\[ = \sum_{n=0}^{\infty} \left[ \mathcal{H} \left( q, \frac{\partial \mathcal{W}_n}{\partial q}, \nu \right) + \frac{\partial \mathcal{W}_n}{\partial \nu} \right] \]
Separating the expanded HJE into terms of equal power in $\epsilon$ yields the series of ordered differential equations

\[
\mathcal{K}_0(q, \hat{p}, \nu) = \mathcal{H}_0(q, \hat{p}, \nu)
\]

\[
\mathcal{K}_1(q, \hat{p}, \nu) = \mathcal{H}_1(q, \hat{p}, \nu) + \frac{\partial \mathcal{W}_1}{\partial \nu}
\]

\[
\mathcal{K}_2(q, \hat{p}, \nu) = \mathcal{H}_2(q, \hat{p}, \nu) + \frac{\partial \mathcal{W}_1}{\partial q} \cdot \frac{\partial \mathcal{H}_1}{\partial p}(q, \hat{p}, \nu) + \frac{\partial \mathcal{W}_2}{\partial \nu}
\]

\[\vdots\]

\[
\mathcal{K}_n(q, \hat{p}, \nu) = \mathcal{Q}_n(q, \hat{p}, \nu) + \frac{\partial \mathcal{W}_n}{\partial \nu}
\]

which term by term, relates $\mathcal{W}_i$ to $\mathcal{K}_i$ through $\mathcal{Q}_i$ known \textit{a priori} (that is, $\mathcal{Q}_i$ depends only on the original Hamiltonian function and terms of lesser order in $\epsilon$).
Outline

5. Elliptic Restricted Three-Body Problem

6. Perturbation Theory
   - HJE & Von Zeipel’s Method

7. DH Single Parameter Method
   - DH Method

8. Extended DH Method Proofs

9. DH Transformation of the ERTBP

10. DH Method Control

11. Nonlinear Oscillator with Damping
Two major disadvantages to Von Zeipel’s method:

- Not canonically invariant - transformation equations change for different sets of canonical variables.
- Form of the generating function $\mathcal{W}(q, \hat{p}, \nu)$ is mixed in terms of original and transformed variables - results in mixed state transformation equations that must be inverted.

In the 1960s, Deprit and Hori each independently derived an alternative method that constructs the ordered differential equations using the Lie transform

\[
\mathcal{L}_{\mathcal{W}\mathcal{H}} \triangleq [\mathcal{H}, \mathcal{W}] = \sum_{i=1}^{N} \left( \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial \mathcal{W}}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial \mathcal{W}}{\partial q_i} \right)
\]
The DH method starts with a nearly-integrable Hamiltonian system

\[ \mathcal{H}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{H}_n^{(0)}(q, p, \nu) \]

expanded about the single system parameter \( \epsilon \), which is transformed by the near-identity transformation

\[
q = Q(\hat{q}, \hat{p}, \epsilon, \nu) \\
p = P(\hat{q}, \hat{p}, \epsilon, \nu)
\]

defined implicitly through the expanded generating function

\[
\mathcal{W}(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{W}_{n+1}(\hat{q}, \hat{p}, \nu)
\]

which depends only on the transformed variables.
DH Method - Single Parameter

The transformed Hamiltonian function is constructed term by term in the series

\[
\mathcal{K}(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{K}_n(\hat{q}, \hat{p}, \nu)
\]

\[
= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \left( H_0^{(n)}(\hat{q}, \hat{p}, \nu) + R_0^{(n)}(\hat{q}, \hat{p}, \nu) \right)
\]

\[
= \hat{H}(\hat{q}, \hat{p}, \epsilon, \nu) + R(\hat{q}, \hat{p}, \epsilon, \nu)
\]

where \( \hat{H} \) represents the original Hamiltonian function written explicitly in terms of the state transformation equations

\[
\hat{H}(\hat{q}, \hat{p}, \epsilon, \nu) = H\left( Q(\hat{q}, \hat{p}, \epsilon, \nu), P(\hat{q}, \hat{p}, \epsilon, \nu), \epsilon, \nu \right)
\]

and \( R \) is a remainder function.
DH Method - Single Parameter

The terms appearing in the transformed Hamiltonian function

\[ K(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \left( H_0^{(n)}(\hat{q}, \hat{p}, \nu) + R_0^{(n)}(\hat{q}, \hat{p}, \nu) \right) \]

are constructed starting in terms of the generating function \( \mathcal{W} \) using Lie transforms

\[ H_n^{(r+1)} = H_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\mathcal{W}_{i+1}} H_{n-i}^{(r)} \]

\[ S_n^{(0)} = -\frac{\partial}{\partial \nu} \mathcal{W}_{n+1} \]

\[ S_n^{(r+1)} = S_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\mathcal{W}_{i+1}} S_{n-i}^{(r)} \]

\[ R_0^{(n)} = S_0^{(n-1)} \]
DH Method - Deprit’s Triangle

Deprit’s Triangle:

Left-most vertical = original Hamiltonian function under the identity transformation \( \mathcal{H}(\hat{q}, \hat{p}, \epsilon, \nu) \)

Right-most diagonal = transformed Hamiltonian \( \hat{\mathcal{H}} \)

\[
\mathcal{H}_n^{(r)} = \mathcal{H}_{n+1}^{(r-1)} + \sum_{m=0}^{n} \binom{n}{m} \mathcal{L} \mathcal{W}_{m+1} \mathcal{H}_{n-m}^{(r-1)}
\]
DH Method - Ordered Differential Equations

Resulting in a series of differential equations for each order of $\epsilon$, all in the form of the homological equation

$$\frac{\partial W_i}{\partial \nu} - \mathcal{L}_{W_i} \mathcal{H}_0^{(0)} = Q_i - K_i$$

For example,

$$K_0 = \mathcal{H}_0^{(0)}$$

$$K_1 = -\frac{\partial W_1}{\partial \nu} + \mathcal{L}_{W_1} \mathcal{H}_0^{(0)} + \mathcal{H}_1^{(0)}$$

$$K_2 = -\frac{\partial W_2}{\partial \nu} + \mathcal{L}_{W_2} \mathcal{H}_0^{(0)} + \mathcal{H}_2^{(0)} + \mathcal{L}_{W_1} \left( \mathcal{H}_1^{(0)} + K_1 \right)$$

$$K_3 = -\frac{\partial W_3}{\partial \nu} + \mathcal{L}_{W_3} \mathcal{H}_0^{(0)} + \mathcal{H}_3^{(0)} + \mathcal{L}_{W_2} \left( 2\mathcal{H}_1^{(0)} + K_1 \right) + \mathcal{L}_{W_1} \left( \mathcal{H}_2^{(0)} + 2K_2 - \mathcal{L}_{W_1} K_1 \right)$$
DH Method - State Transformation Equations

The expansion of the explicit state transformation equations $q = Q(\hat{q}, \hat{p}, \epsilon, \nu)$ and $p = P(\hat{q}, \hat{p}, \epsilon, \nu)$ is represented by

$$
q = Q(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} q_0^{(n)}(\hat{q}, \hat{p}, \nu)
$$

$$
p = P(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} p_0^{(n)}(\hat{q}, \hat{p}, \nu)
$$

and may be constructed using the recursive equations

$$
q_n^{(r+1)} = q_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} L_{W_{i+1}} q_{n-i}^{(r)}
$$

$$
p_n^{(r+1)} = p_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} L_{W_{i+1}} p_{n-i}^{(r)}
$$

where $q_0^{(0)} = \hat{q}$, $p_0^{(0)} = \hat{p}$, and $q_n^{(0)} = p_n^{(0)} = 0$ for $n > 0$. 
Moreover, the inverse transformation

\[ \hat{q} = \hat{Q}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \hat{q}_0^{(n)}(q, p, \nu) \]

\[ \hat{p} = \hat{P}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \hat{p}_0^{(n)}(q, p, \nu) \]

is obtained by defining the inverse generating function

\[ \hat{\mathcal{W}} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \hat{\mathcal{W}}_0^{(n)} \]

with \( \hat{\mathcal{W}}_n^{(0)} = -\mathcal{W}_n \) and constructed using the recursive equation

\[ \hat{\mathcal{W}}_n^{(r+1)} = \hat{\mathcal{W}}_n^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\mathcal{W}_{i+1}} \hat{\mathcal{W}}_{n-i}^{(r)} \]
and applying the recursive equations

\[ \hat{q}_n^{(r+1)} = \hat{q}_n^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\tilde{W}_{i+1}} \hat{q}_{n-i}^{(r)} \]

\[ \hat{p}_n^{(r+1)} = \hat{p}_n^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\tilde{W}_{i+1}} \hat{p}_{n-i}^{(r)} \]

where \( \hat{q}_0^{(0)} = q \), \( \hat{p}_0^{(0)} = p \), and \( \hat{q}_n^{(0)} = \hat{p}_n^{(0)} = 0 \) for \( n > 0 \). The inverse equations

\[ \hat{q} = \hat{Q}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \hat{q}_0^{(n)}(q, p, \nu) \]

\[ \hat{p} = \hat{P}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \hat{p}_0^{(n)}(q, p, \nu) \]
Outline

5. Elliptic Restricted Three-Body Problem

6. Perturbation Theory
   - HJE & Von Zeipel’s Method

7. DH Single Parameter Method
   - DH Method

8. Extended DH Method Proofs

9. DH Transformation of the ERTBP

10. DH Method Control

11. Nonlinear Oscillator with Damping
The canonical transformation \( q = Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \) and \( p = P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \) is derived from the pair of non-autonomous systems

\[
\frac{\partial}{\partial \epsilon} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial p} \mathcal{W}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \\ -\frac{\partial}{\partial q} \mathcal{W}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \end{pmatrix}
\]

and

\[
\frac{\partial}{\partial \gamma} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial p} \mathcal{V}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \\ -\frac{\partial}{\partial q} \mathcal{V}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \end{pmatrix}
\]
Define the extended phase space by appending the independent variable $\nu$ and the Hamiltonian function to the original state variables

$$ q \rightarrow \{ q, \nu \} $$

$$ p \rightarrow \{ p, -\mathcal{H} \} $$

To apply the Lie operator within the extended phase space but still maintain its definition within the original $(q, p)$ phase space, one must incorporate the additional terms

$$ \mathcal{L}_W - \frac{\partial W}{\partial \mathcal{H}} \frac{\partial}{\partial \nu} + \frac{\partial W}{\partial \nu} \frac{\partial}{\partial \mathcal{H}} = \mathcal{L}_W + \frac{\partial W}{\partial \nu} \frac{\partial}{\partial \mathcal{H}} $$

$$ \mathcal{L}_V - \frac{\partial V}{\partial \mathcal{H}} \frac{\partial}{\partial \nu} + \frac{\partial V}{\partial \nu} \frac{\partial}{\partial \mathcal{H}} = \mathcal{L}_V + \frac{\partial V}{\partial \nu} \frac{\partial}{\partial \mathcal{H}} $$
The Deprit operators are defined as

\[
\mathcal{D}_W \triangleq \frac{\partial}{\partial \epsilon} + \mathcal{L}_W
\]

\[
\mathcal{D}_V \triangleq \frac{\partial}{\partial \gamma} + \mathcal{L}_V
\]

and incorporating the additional terms for the expanded phase space, the extended Deprit operators are defined as

\[
\mathcal{E}_W \triangleq \frac{\partial}{\partial \epsilon} + \mathcal{L}_W + \frac{\partial W}{\partial \nu} \frac{\partial}{\partial H} = \mathcal{D}_W + \frac{\partial W}{\partial \nu} \frac{\partial}{\partial H}
\]

\[
\mathcal{E}_V \triangleq \frac{\partial}{\partial \gamma} + \mathcal{L}_V + \frac{\partial V}{\partial \nu} \frac{\partial}{\partial H} = \mathcal{D}_V + \frac{\partial V}{\partial \nu} \frac{\partial}{\partial H}
\]
The partial derivative of $F(q, p, \epsilon, \gamma, \nu)$ with respect to $\epsilon$ is

$$\frac{\partial}{\partial \epsilon} \left( F(q, p, \epsilon, \gamma, \nu) \bigg|_{q=Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)} \right)$$

$$= \left( \frac{\partial F}{\partial \epsilon} + \frac{\partial F}{\partial q} \frac{\partial q}{\partial \epsilon} + \frac{\partial F}{\partial p} \frac{\partial p}{\partial \epsilon} \right) \bigg|_{q=Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)}$$

$$= \left( \frac{\partial F}{\partial \epsilon} + \frac{\partial F}{\partial q} \frac{\partial \mathcal{W}}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial \mathcal{W}}{\partial q} \right) \bigg|_{q=Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)}$$

$$= \left( \frac{\partial F}{\partial \epsilon} + \mathcal{L}_{\mathcal{W}} F \right) \bigg|_{q=Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)}$$

$$= \left( \mathcal{D}_{\mathcal{W}} F \right) \bigg|_{q=Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)}$$
DH Method - Proof

By the same logic, the partial derivative with respect to $\gamma$ is

$$\frac{\partial}{\partial \gamma} \left( F(q, p, \epsilon, \gamma, \nu) \bigg| \begin{array}{l} q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu) \end{array} \right) = \left( \mathcal{D}_\gamma F \right) \bigg| \begin{array}{l} q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu) \end{array}$$

and by extension, the mixed $\epsilon$- and $\gamma$-derivatives to any order are

$$\frac{\partial^n}{\partial \epsilon^n} \frac{\partial^m}{\partial \gamma^m} \left( F(q, p, \epsilon, \gamma, \nu) \bigg| \begin{array}{l} q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu) \end{array} \right) = \left( \mathcal{D}_\epsilon^n \mathcal{D}_\gamma^m F \right) \bigg| \begin{array}{l} q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu) \end{array}$$
Consider the canonical transformation of a non-autonomous Hamiltonian function resulting in a transformed Hamiltonian function in the form

\[ K = \mathcal{H}\left(Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \right) + \mathcal{R}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \]

\[ = \hat{\mathcal{H}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) + \mathcal{R}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \]

where \( \hat{\mathcal{H}} \) represents the original Hamiltonian function written explicitly in terms of the state transformation equations and \( \mathcal{R} \) represents a remainder function.
The expansion of $\hat{H}$ in a Taylor series about $\epsilon = 0$ and $\gamma = 0$ is

$$\hat{H}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \partial^n \partial_m \hat{H}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \bigg|_{\epsilon, \gamma=0}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \partial^n \partial_m \left( \mathcal{H}(q, p, \epsilon, \gamma, \nu) \bigg|_{q=Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)} \bigg|_{p=P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)} \right) \bigg|_{\epsilon, \gamma=0}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \left( \mathcal{D}_V^n \mathcal{D}_V^m \mathcal{H}(q, p, \epsilon, \gamma, \nu) \bigg|_{q=Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)} \bigg|_{p=P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)} \right) \bigg|_{\epsilon, \gamma=0}$$
DH Method - Proof

Introducing the subscripted and superscripted formulation

\[ \mathcal{H} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} H^{(0,0)}_{m,n} \]

\[ D_W^s D_V^r \mathcal{H} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \mathcal{H}^{(r,s)}_{m,n} \]

the expansion of the original Hamiltonian function is represented by the series of functions \( \mathcal{H}^{(0,0)}_{m,n} \) and the expansion of \( \hat{\mathcal{H}} \) is represented by

\[ \hat{\mathcal{H}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \left( D_W^n D_V^m \mathcal{H}(q, p, \epsilon, \gamma, \nu) \right) \left| \begin{array}{c} q = Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \\ p = P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \end{array} \right|_{\epsilon, \gamma = 0} \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} H^{(m,n)}_{0,0}(\hat{q}, \hat{p}, \nu) \]
DH Method - Proof

Within the subscripted and superscripted formulation, the Lie derivatives satisfy

\[ \mathcal{L}_\mathcal{W} \mathcal{H}^{(r,s)}_{m,n} = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{H}^{(r,s)}_{m-j,n-i} \]

\[ \mathcal{L}_\mathcal{V} \mathcal{H}^{(r,s)}_{m,n} = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \mathcal{H}^{(r,s)}_{m-j,n-i} \]
such that the terms included in the expansion series under the Deprit operator satisfy

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \mathcal{H}^{(r,s+1)}_{m,n} = \mathcal{D}_\mathcal{W} \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \mathcal{H}^{(r,s)}_{m,n} \right)
\]

\[
\frac{\partial}{\partial \epsilon} \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \mathcal{H}^{(r,s)}_{m,n} \right) + \mathcal{L}_\mathcal{W} \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \mathcal{H}^{(r,s)}_{m,n} \right)
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \left( \mathcal{H}^{(r,s)}_{m,n+1} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_\mathcal{W}_{j+1,i+1} \mathcal{H}^{(r,s)}_{m-j,n-i} \right)
\]

and likewise for \( \mathcal{D}_\mathcal{V} \).
DH Method - Proof

Thus, all the unknown functions $\mathcal{H}^{(r,s)}_{m,n}$ may be constructed term by term using recursive equations

$$
\mathcal{H}^{(r,s+1)}_{m,n} = \mathcal{H}^{(r,s)}_{m,n} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{j+1,i+1} \mathcal{H}^{(r,s)}_{m-j,n-i}
$$

$$
\mathcal{H}^{(r+1,s)}_{m,n} = \mathcal{H}^{(r,s)}_{m+1,n} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{j+1,i+1} \mathcal{H}^{(r,s)}_{m-j,n-i}
$$
DH Method - Proof

Applying the extended Deprit operators to the Hamiltonian state variable yields

\[
\mathcal{E}_W \mathcal{H} = D_W \mathcal{H} - \frac{\partial W}{\partial \nu}
\]
\[
\mathcal{E}_V \mathcal{H} = D_V \mathcal{H} - \frac{\partial V}{\partial \nu}
\]

and for higher-order and mixed terms,

\[
\mathcal{E}_W^s \mathcal{E}_V^r \mathcal{H} = D_W^s D_V^r \mathcal{H} - D_W^{s-1} D_V^r \frac{\partial W}{\partial \nu} - D_W^s D_V^{r-1} \frac{\partial V}{\partial \nu}
\]

The first term represents the autonomous part of the transformation \( \hat{\mathcal{H}} \) while the other two terms comprise the remainder function.
Define the intermediary functions $S$ and $T$ by

$$S_{m,n}^{(0,0)} = -\frac{\partial}{\partial \nu} \mathcal{W}_{m+1,n+1}$$

$$T_{m,n}^{(0,0)} = -\frac{\partial}{\partial \nu} \mathcal{V}_{m+1,n+1}$$

such that the remainder function $\mathcal{R}$ is constructed term by term according to

$$\mathcal{R}_{0,0}^{(m,n)} = \begin{cases} 
S_{0,0}^{(0,n-1)} & m = 0, n \neq 0 \\
T_{0,0}^{(m-1,0)} & m \neq 0, n = 0 \\
S_{0,0}^{(m,n-1)} = T_{0,0}^{(m-1,n)} & m, n \neq 0 
\end{cases}$$
DH Method - Proof

and the recursive formulae

\[
S_{m,n}^{(r,s+1)} = S_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{j+1,i+1} S_{m-j,n-i}^{(r,s)}
\]

\[
S_{m,n}^{(r+1,s)} = S_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{j+1,i+1} S_{m-j,n-i}^{(r,s)}
\]

and

\[
T_{m,n}^{(r,s+1)} = T_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{j+1,i+1} T_{m-j,n-i}^{(r,s)}
\]

\[
T_{m,n}^{(r+1,s)} = T_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{j+1,i+1} T_{m-j,n-i}^{(r,s)}
\]

QED
Based on the preceding discussion, one may derive either of

\[ D_W D_V \mathcal{H} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \mathcal{H}_{m,n}^{(1,1)} \]

\[ D_V D_W \mathcal{H} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \mathcal{H}_{m,n}^{(1,1)} \]

For consistency’s sake, the terms on the right-hand side should be the same in either case, which implies that the Deprit operators are commutative

\[ D_W D_V = D_V D_W \]

The Lie derivatives are not themselves commutative, but instead satisfy the condition

\[ \mathcal{L}_V \mathcal{L}_W = \mathcal{L}_W \mathcal{L}_V - \mathcal{L}_{\mathcal{L}_W V} \]
DH Method - Proof of Commutativity

Expanding the Deprit commutative condition yields

\[ \mathcal{D}_W \mathcal{D}_V = \frac{\partial}{\partial \epsilon} \frac{\partial}{\partial \gamma} + \mathcal{L}_V + \mathcal{L}_W \frac{\partial}{\partial \gamma} + \mathcal{L}_W \mathcal{L}_V \]

\[ = \mathcal{D}_V \mathcal{D}_W = \frac{\partial}{\partial \gamma} \frac{\partial}{\partial \epsilon} + \mathcal{L}_W + \mathcal{L}_V \frac{\partial}{\partial \epsilon} + \mathcal{L}_V \mathcal{L}_W \]

Substituting in the Lie commutation condition yields

\[ 0 = \frac{\partial}{\partial \epsilon} \mathcal{L}_V + \mathcal{L}_W \frac{\partial}{\partial \gamma} - \frac{\partial}{\partial \gamma} \mathcal{L}_W - \mathcal{L}_V \frac{\partial}{\partial \epsilon} + \mathcal{L}_V \mathcal{L}_W \]

Moreover, the partial derivatives and Lie operators satisfy the conditions

\[ \frac{\partial}{\partial \epsilon} \mathcal{L}_V - \mathcal{L}_V \frac{\partial}{\partial \epsilon} = \mathcal{L}_{\partial V/\partial \epsilon} \]

\[ \frac{\partial}{\partial \gamma} \mathcal{L}_W - \mathcal{L}_W \frac{\partial}{\partial \gamma} = \mathcal{L}_{\partial W/\partial \gamma} \]
DH Method - Proof of Commutativity

Together these yield the condition

\[ 0 = \mathcal{L} \frac{\partial \mathcal{V}}{\partial \epsilon} - \mathcal{L} \frac{\partial \mathcal{W}}{\partial \gamma} + \mathcal{L} \mathcal{W} \mathcal{V} \]

which implies

\[ 0 = \frac{\partial \mathcal{V}}{\partial \epsilon} - \frac{\partial \mathcal{W}}{\partial \gamma} + \mathcal{L} \mathcal{W} \mathcal{V} \]

This condition is herein referred to as the Deprit commutation condition. By applying the recursive algorithms used previously, the Deprit commutation condition can also be expressed term by term as

\[ 0 = \mathcal{V}_{m+1,n+2} - \mathcal{W}_{m+2,n+1} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L} \mathcal{W}_{j+1,i+1} \mathcal{V}_{m-j+1,n-i+1} \]

QED
Consider the explicit state transformation equations
\[ q = Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \] and
\[ p = P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \]
expanded about \( \epsilon = 0 \) and \( \gamma = 0 \)

\[
q = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! \ n!} \frac{\partial^n}{\partial \epsilon^n} \frac{\partial^m}{\partial \gamma^m} \left( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \right) \bigg|_{\epsilon, \gamma=0}
\]

\[
p = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! \ n!} \frac{\partial^n}{\partial \epsilon^n} \frac{\partial^m}{\partial \gamma^m} \left( P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \right) \bigg|_{\epsilon, \gamma=0}
\]

The expansion series are represented in terms of the Deprit operators by

\[
q = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! \ n!} \left( D^n_{\hat{W}} D^m_{\hat{V}} Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \right) \bigg|_{\epsilon, \gamma=0}
\]

\[
p = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! \ n!} \left( D^n_{\hat{W}} D^m_{\hat{V}} P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \right) \bigg|_{\epsilon, \gamma=0}
\]
DH Method - Proof of State Transformation Eqns

The subscripted and superscripted formulation is defined as before such that the expanded state transformation equations are

\[
q = Q(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! \ n!} q_{0,0}^{(m,n)} (\hat{q}, \hat{p}, \nu)
\]

\[
p = P(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! \ n!} p_{0,0}^{(m,n)} (\hat{q}, \hat{p}, \nu)
\]

and the zero superscript terms correspond to the identity transformation,

\[
q_{m,n}^{(0,0)} = \begin{cases} 
\hat{q} & m = n = 0 \\
0 & m + n > 0 
\end{cases}
\]

\[
p_{m,n}^{(0,0)} = \begin{cases} 
\hat{p} & m = n = 0 \\
0 & m + n > 0 
\end{cases}
\]
Since the state transformation equations are not explicitly dependent on the Hamiltonian function as a state variable, they transform in the same manner as $\hat{h}$, namely,

$$q^{(r,s+1)}_{m,n} = q^{(r,s)}_{m,n+1} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} L^{W_{j+1,j+1}} q^{(r,s)}_{m-j,n-i}$$

$$q^{(r+1,s)}_{m,n} = q^{(r,s)}_{m+1,n} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} L^{V_{j+1,j+1}} q^{(r,s)}_{m-j,n-i}$$

and

$$p^{(r,s+1)}_{m,n} = p^{(r,s)}_{m,n+1} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} L^{W_{j+1,j+1}} p^{(r,s)}_{m-j,n-i}$$

$$p^{(r+1,s)}_{m,n} = p^{(r,s)}_{m+1,n} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} L^{V_{j+1,j+1}} p^{(r,s)}_{m-j,n-i}$$

QED
The inversion of the canonical transformation, defined as

$$\hat{q} = \hat{Q}(q, p, \epsilon, \gamma, \nu)$$

$$\hat{p} = \hat{P}(q, p, \epsilon, \gamma, \nu)$$

satisfies the system equations

$$\frac{\partial}{\partial \epsilon} \left( \begin{array}{c} \hat{q} \\ \hat{p} \end{array} \right) = \left( \begin{array}{c} \frac{\partial}{\partial \hat{p}} \hat{W}(q, p, \epsilon, \gamma, \nu) \\ -\frac{\partial}{\partial \hat{q}} \hat{W}(q, p, \epsilon, \gamma, \nu) \end{array} \right)$$

and

$$\frac{\partial}{\partial \gamma} \left( \begin{array}{c} \hat{q} \\ \hat{p} \end{array} \right) = \left( \begin{array}{c} \frac{\partial}{\partial \hat{p}} \hat{V}(q, p, \epsilon, \gamma, \nu) \\ -\frac{\partial}{\partial \hat{q}} \hat{V}(q, p, \epsilon, \gamma, \nu) \end{array} \right)$$

in terms of the inverse generating functions $\hat{W}(q, p, \epsilon, \gamma, \nu)$ and $\hat{V}(q, p, \epsilon, \gamma, \nu)$. 
The remainder function for the inverse transformation 
\((q, p) \leftarrow (\hat{q}, \hat{p})\) is [Winter, 1941]

\[
\hat{R} = -\mathcal{R}\left( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \right)
\]

implying that the inverse generating functions are

\[
\hat{\mathcal{W}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = -\mathcal{W}\left( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \right)
\]

\[
\hat{V}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = -\mathcal{V}\left( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \right)
\]

Thus, to construct the inverse generating functions under the canonical transformation, one need only apply the DH method in the same manner as in the transformation of the Hamiltonian function.
The subscript and superscript formulation is introduced such that

\[
\hat{W} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! \ n!} \hat{W}_{0,0}^{(m,n)}(q, p, \nu)
\]

\[
\hat{V} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! \ n!} \hat{V}_{0,0}^{(m,n)}(q, p, \nu)
\]

and the zeroth-order terms correspond to the negative identity transformation

\[
\hat{W}_{m,n}^{(0,0)} = -W_{m,n}(\hat{q}, \hat{p}, \nu)
\]

\[
\hat{V}_{m,n}^{(0,0)} = -V_{m,n}(\hat{q}, \hat{p}, \nu)
\]
The transformed inverse generating functions are then constructed through the DH recursive equations

\[
\hat{W}_{m,n}^{(r,s+1)} = \hat{W}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{j+1,i+1} \hat{W}_{m-j,n-i}^{(r,s)}
\]

\[
\hat{W}_{m,n}^{(r+1,s)} = \hat{W}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{j+1,i+1} \hat{W}_{m-j,n-i}^{(r,s)}
\]

The inverse transformation equations are represented by

\[
\hat{q} = \hat{Q}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \hat{q}^{(m,n)}_{0,0}(q, p, \nu)
\]

\[
\hat{p} = \hat{P}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \hat{p}^{(m,n)}_{0,0}(q, p, \nu)
\]

with \(\hat{q}_{0,0}^{(0,0)} = q, \hat{p}_{0,0}^{(0,0)} = p, \) and \(\hat{q}_{m,n}^{(0,0)} = \hat{p}_{m,n}^{(0,0)} = 0\) for \(m + n > 0\)
and are generated through the recursive equations

\[ \hat{q}_{m,n}^{(r,s+1)} = \hat{q}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\hat{W}_{j+1,i+1}}^{i,j} \hat{q}_{m-j,n-i}^{(r,s)} \]

\[ \hat{q}_{m,n}^{(r+1,s)} = \hat{q}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\hat{V}_{j+1,i+1}}^{i,j} \hat{q}_{m-j,n-i}^{(r,s)} \]

\[ \hat{p}_{m,n}^{(r,s+1)} = \hat{p}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\hat{W}_{j+1,i+1}}^{i,j} \hat{p}_{m-j,n-i}^{(r,s)} \]

\[ \hat{p}_{m,n}^{(r+1,s)} = \hat{p}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\hat{V}_{j+1,i+1}}^{i,j} \hat{p}_{m-j,n-i}^{(r,s)} \]

QED
Outline

5. Elliptic Restricted Three-Body Problem

6. Perturbation Theory
   - HJE & Von Zeipel’s Method

7. DH Single Parameter Method
   - DH Method

8. Extended DH Method Proofs

9. DH Transformation of the ERTBP

10. DH Method Control

11. Nonlinear Oscillator with Damping
Based on the form of $\mathcal{H}_{0,0}^{(0,0)}$, the homological equation is

$$\left( \frac{\partial}{\partial \nu} + \omega_s \frac{\partial}{\partial \theta_s} - \omega_\ell \frac{\partial}{\partial \theta_\ell} + \omega_z \frac{\partial}{\partial \theta_z} \right) \mathcal{W}_{i+1,j} = Q_{i,j} - K_{i,j}$$

for which the transformed Hamiltonian function is defined by the periodic average

$$K_{i,j} \overset{\Delta}{=} \langle Q_{i,j} \rangle = \frac{1}{4(2\pi)^4} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} Q_{i,j} \, d\nu \, d\theta_s \, d\theta_\ell \, d\theta_z$$

such that $K_{i,j} = K_{i,j}(\hat{I})$ under the transformation $(\theta, I) \rightarrow (\hat{\theta}, \hat{I})$. 
DH Method - Inversion of the Homological Equations

For each ordered homological equation, $\mathcal{K}_{m,n}$ is prescribed by the average

$$
\mathcal{K}_{i,j} = \overline{Q_{i,j}} = \frac{1}{4(2\pi)} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} Q_{i,j} \, d\nu \, d\theta_s \, d\theta_\ell \, d\theta_z.
$$

and the corresponding generating function is derived from the substitution

$$
\cos (i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z) \rightarrow \frac{\sin (i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z)}{i_1 + i_2 \omega_s - i_3 \omega_\ell + i_4 \omega_z}
$$

$$
\sin (i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z) \rightarrow -\frac{\cos (i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z)}{i_1 + i_2 \omega_s - i_3 \omega_\ell + i_4 \omega_z}.
$$
Numerical Validation - Test Cases

Near-Circular Earth-Moon
($\mu = 0.0124$, $e = 0.0549$):

Eccentric Earth-Moon
($\mu = 0.0124$, $e = 0.2$):

Figure: Poincaré Surface of Section near Earth-Moon $L_5$

Figure: Poincaré Surface of Section near Eccentric $L_5$
Numerical Validation - Earth-Moon System

(a) In-Plane Trajectory

(b) Out-of-Plane Trajectory

(c) Cartesian Phase Portrait
   (Red = $x$, Green = $y$, Yellow = $z$)

(d) SLP Phase Portrait
   (Cyan = $s$, Brown = $\ell$, Yellow = $z$)
Numerical Validation - Earth-Moon Convergence

Figure: DH Convergence for Earth-Moon System
Numerical Validation - Earth-Moon System

Convergence

Figure: DH Convergence for Earth-Moon System
Numerical Validation - Eccentric Earth-Moon System

(a) In-Plane Trajectory

(b) Out-of-Plane Trajectory

(c) Cartesian Phase Portrait
   (Red = x, Green = y, Yellow = z)

(d) SLP Phase Portrait
   (Cyan = s, Brown = ℓ, Yellow = z)
Numerical Validation - Eccentric E-M Convergence

Figure: DH Convergence for Eccentric Earth-Moon System
Numerical Validation - Eccentric Earth-Moon System Convergence

Figure: DH Convergence for Eccentric Earth-Moon System
Outline

5 Elliptic Restricted Three-Body Problem

6 Perturbation Theory
   • HJE & Von Zeipel’s Method

7 DH Single Parameter Method
   • DH Method

8 Extended DH Method Proofs

9 DH Transformation of the ERTBP

10 DH Method Control

11 Nonlinear Oscillator with Damping
DH Method - Control

Append the original Hamiltonian function with a non-autonomous forcing function \( U(q, p, \epsilon, \gamma, \nu) \)

\[
\mathcal{H}_c(q, p, \epsilon, \gamma, \nu) = \mathcal{H}(q, p, \epsilon, \gamma, \nu) + U(q, p, \epsilon, \gamma, \nu)
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m! n!} \left( \mathcal{H}^{(0,0)}_{m,n}(q, p, \nu) + U^{(0,0)}_{m,n}(q, p, \nu) \right)
\]

yielding

\[
\mathcal{K}_c = \mathcal{H}_c \left( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \right) + \mathcal{R}_c (\hat{q}, \hat{p}, \epsilon, \gamma, \nu)
\]

\[
= \mathcal{H} \left( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \right)
\]

\[
+ U \left( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \right) + \mathcal{R}_c (\hat{q}, \hat{p}, \epsilon, \gamma, \nu)
\]
DH Method - Control

The controlled homological equation is

\[
\frac{\partial \mathcal{W}_{c,i+1,j}}{\partial \nu} - \mathcal{L} \mathcal{W}_{c,i+1,j} \mathcal{H}_{c,0,0}^{(0,0)} = Q_{c,i,j} - K_{c,i,j}
\]

where \( Q_{c,i,j} \) is known \textit{a priori} and \( K_{c,i,j} \) is the transformed controlled Hamiltonian.

Zero control is applied to the unperturbed system

\[
U_{0,0}^{(0,0)} \equiv 0 \quad \implies \quad \mathcal{H}_{c,0,0}^{(0,0)} = \mathcal{H}_{0,0}^{(0,0)}
\]

and the controlled generating functions are de-coupled into a sum of independent generating functions

\[
\mathcal{W}_c = \mathcal{W} + \mathcal{W}_u
\]

\[
\mathcal{V}_c = \mathcal{V} + \mathcal{V}_u
\]
DH Method - Control

The controlled homological equation then takes either form

\[ K_{c,i,j} = -\frac{\partial W_{i+1,j}}{\partial \nu} + \mathcal{L} W_{i+1,j} H_{0,0}^{(0,0)} + Q_{i,j} \]

\[ \quad - \frac{\partial W_{u,i+1,j}}{\partial \nu} + \mathcal{L} W_{u,i+1,j} H_{0,0}^{(0,0)} + Q_{u,i,j} \]

\[ \quad + U_{i,j}^{(0,0)} + Q_{c,i,j} \]

where \( Q_{i,j}, Q_{u,i,j} \) and \( Q_{c,i,j} \) are known \textit{a priori} based on the original Hamiltonian function and terms of lesser order.
The controlled homological equation then takes either form

\[ K_{c,i,j} = -\frac{\partial \mathcal{W}_{i+1,j}}{\partial \nu} + \mathcal{L}\mathcal{W}_{i+1,j} \mathcal{H}_{0,0}^{(0,0)} + Q_{i,j} = K_{i,j} \]

\[ -\frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L}\mathcal{W}_{u,i+1,j} \mathcal{H}_{0,0}^{(0,0)} + Q_{u,i,j} \]

\[ + \mathcal{U}_{i,j}^{(0,0)} + Q_{c,i,j} \]

**General Control Law**

\[ K_{c,i,j} - K_{i,j} = -\frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L}\mathcal{W}_{u,i+1,j} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{U}_{i,j}^{(0,0)} \]

\[ + Q_{u,i,j} + Q_{c,i,j} \]

where \( \mathcal{U}_{i,j}^{(0,0)} \) and \( \mathcal{W}_{u,i+1,j} \) (likewise \( \mathcal{V}_{u,i,j+1} \)) are the control functions designed to achieve a desired form for \( K_{c,i,j} \).
\[ \mathcal{K}_{c,0,0} = \mathcal{H}^{(0,0)}_{0,0} = \mathcal{K}_{0,0} \]

\[ \mathcal{K}_{c,0,1} = \mathcal{H}^{(0,1)}_{c,0,0} + \mathcal{S}^{(0,0)}_{c,0,0} = \mathcal{K}_{0,1} + \mathcal{U}^{(0,0)}_{0,1} - \frac{\partial \mathcal{W}_{u,1,1}}{\partial \nu} + \mathcal{L} \mathcal{W}_{u,1,1} \mathcal{H}^{(0,0)}_{0,0} \]

\[ \mathcal{K}_{c,1,0} = \mathcal{H}^{(1,0)}_{c,0,0} + \mathcal{T}^{(0,0)}_{c,0,0} = \mathcal{K}_{1,0} + \mathcal{U}^{(0,0)}_{1,0} - \frac{\partial \mathcal{V}_{u,1,1}}{\partial \nu} + \mathcal{L} \mathcal{V}_{u,1,1} \mathcal{H}^{(0,0)}_{0,0} \]

\[ \mathcal{K}_{c,0,2} = \mathcal{H}^{(0,2)}_{c,0,0} + \mathcal{S}^{(0,1)}_{c,0,0} = \mathcal{K}_{0,2} + \mathcal{U}^{(0,0)}_{0,2} - \frac{\partial \mathcal{W}_{u,1,2}}{\partial \nu} + \mathcal{L} \mathcal{W}_{u,1,2} \mathcal{H}^{(0,0)}_{0,0} \]

\[ + \mathcal{L} \mathcal{W}_{1,1} \left( \mathcal{U}^{(0,0)}_{0,1} + \mathcal{U}^{(0,1)}_{0,0} + \mathcal{S}^{(0,0)}_{u,0,0} \right) \]

\[ + \mathcal{L} \mathcal{W}_{u,1,1} \left( \mathcal{H}^{(0,0)}_{c,0,1} + \mathcal{H}^{(0,1)}_{c,0,0} + \mathcal{S}^{(0,0)}_{c,0,0} \right) \]

\[ \vdots \]

\[ \mathcal{K}_{c,i,j} = \mathcal{H}^{(i,j)}_{c,0,0} + \mathcal{S}^{(i,j-1)}_{c,0,0} = \mathcal{K}_{i,j} + \mathcal{U}^{(0,0)}_{i,j} - \frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L} \mathcal{W}_{u,i+1,j} \mathcal{H}^{(0,0)}_{0,0} \]

\[ + \mathcal{Q}_{u,i,j} + \mathcal{Q}_{c,i,j} \]
DH Method - Secular Control

Secular Control

\[ \mathcal{W}_u = \mathcal{V}_u = 0 \]

\[ \mathcal{U}_{i,j}^{(0,0)} = \mathcal{K}_{c,i,j} - \mathcal{K}_{i,j} - \mathcal{Q}_{c,i,j} \]

\[ \mathcal{U}_{0,1}^{(0,0)} = \mathcal{K}_{c,0,1} - \mathcal{K}_{0,1} \]

\[ \mathcal{U}_{1,0}^{(0,0)} = \mathcal{K}_{c,1,0} - \mathcal{K}_{1,0} \]

\[ \mathcal{U}_{0,2}^{(0,0)} = \mathcal{K}_{c,0,2} - \mathcal{K}_{0,2} - \mathcal{L}_{\mathcal{W}_{1,1}} \left( \mathcal{U}_{0,1}^{(0,0)} + \mathcal{U}_{0,0}^{(0,1)} \right) \]

\[ \mathcal{U}_{2,0}^{(0,0)} = \mathcal{K}_{c,2,0} - \mathcal{K}_{2,0} - \mathcal{L}_{\mathcal{V}_{1,1}} \left( \mathcal{U}_{1,0}^{(0,0)} + \mathcal{U}_{0,0}^{(1,0)} \right) \]

\[ \mathcal{U}_{1,1}^{(0,0)} = \mathcal{K}_{c,1,1} - \mathcal{K}_{1,1} - \mathcal{L}_{\mathcal{V}_{1,1}} \mathcal{U}_{0,1}^{(0,0)} - \mathcal{L}_{\mathcal{W}_{1,1}} \mathcal{U}_{0,0}^{(1,0)} \]

\[ = \mathcal{K}_{c,1,1} - \mathcal{K}_{1,1} - \mathcal{L}_{\mathcal{W}_{1,1}} \mathcal{U}_{1,0}^{(0,0)} - \mathcal{L}_{\mathcal{V}_{1,1}} \mathcal{U}_{0,0}^{(0,1)} \]
### DH Method - Direct Control

**Direct Control**

\[
\mathcal{W}_{u,i,j} = -\mathcal{W}_{i,j} \quad \mathcal{V}_{u,i,j} = -\mathcal{V}_{i,j}
\]

\[
\mathcal{U}_{i,j}^{(0,0)} = \mathcal{K}_{c,i,j} - \mathcal{H}_{i,j}^{(0,0)}
\]

\[
\begin{align*}
\mathcal{U}_{0,1}^{(0,0)} &= \mathcal{K}_{c,0,1} - \mathcal{H}_{0,1}^{(0,0)} \\
\mathcal{U}_{1,0}^{(0,0)} &= \mathcal{K}_{c,1,0} - \mathcal{H}_{1,0}^{(0,0)} \\
\mathcal{U}_{0,2}^{(0,0)} &= \mathcal{K}_{c,0,2} - \mathcal{H}_{0,2}^{(0,0)} \\
\mathcal{U}_{2,0}^{(0,0)} &= \mathcal{K}_{c,2,0} - \mathcal{H}_{2,0}^{(0,0)} \\
\mathcal{U}_{1,1}^{(0,0)} &= \mathcal{K}_{c,1,1} - \mathcal{H}_{1,1}^{(0,0)}
\end{align*}
\]

which trivially acts on the higher-order terms of the original Hamiltonian function.
Outline

5. Elliptic Restricted Three-Body Problem

6. Perturbation Theory
   - HJE & Von Zeipel’s Method

7. DH Single Parameter Method
   - DH Method

8. Extended DH Method Proofs

9. DH Transformation of the ERTBP

10. DH Method Control

11. Nonlinear Oscillator with Damping
The standard equation of motion for the nonlinear oscillator with damping is

\[
\frac{d^2q}{dt^2} + \alpha \frac{dq}{dt} + \omega_0^2 \sin q = 0
\]

where \( q \) is the angular displacement, \( \omega_0 \) is the natural, undamped frequency, and \( \alpha \) is the damping coefficient. The dynamics may be expressed in terms of the Hamiltonian function

\[
\mathcal{H}(q, p, t) = e^{-\alpha t} \left( \frac{p^2}{2} - e^{2\alpha t} \omega_0^2 (\cos q - 1) \right)
\]

where \( p = e^{\alpha t} \dot{q} \) is the generalized momenta.[Huang, 2002]
Expanding the resultant Hamiltonian function about the origin and the zero-damping case yields the series representation

\[
\mathcal{H}(q, p, t) = \frac{p^2}{2} \left( 1 - \alpha t + \frac{\alpha^2 t^2}{2} - \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + \mathcal{O}(\alpha^5) \right) \\
+ \omega_0^2 \left( 1 + \alpha t + \frac{\alpha^2 t^2}{2} + \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + \mathcal{O}(\alpha^5) \right) \\
\left( q^2 - \frac{\gamma^2 q^4}{4!} + \frac{\gamma^4 q^6}{6!} - \frac{\gamma^6 q^8}{8!} + \mathcal{O}(\gamma^8) \right)
\]

where \( \gamma \) parameterizes the scale of the even-powered nonlinear terms.
Damped Oscillator

Action-angle type variables are introduced in terms of the canonical transformation generating function

\[ S(q, \theta) = \frac{\omega_0}{2} q^2 \tan^{-1} \theta \]

or directly in terms of the state transformation equations

\[ q = \sqrt{\frac{2I}{\omega_0}} \sin \theta \]
\[ p = \sqrt{2I\omega_0} \cos \theta \]

Expressed in terms of action-angle type variables, the expanded Hamiltonian function is

\[ \mathcal{H}(\theta, l, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\gamma^m \alpha^n}{m! n!} \mathcal{H}^{(0,0)}_{m,n}(\theta, l, t) \]
Damped Oscillator

The unperturbed system is the linearized oscillator with zero damping

\[ H^{(0,0)}_0 = \omega_0 I = \frac{1}{2} \left( p^2 + \omega_0 q^2 \right) \]

wherein the phase space is foliated by invariant tori parameterized by the constant value \( I_0 = I(t_0) \) and wound linearly by \( \theta(t) = \theta(t_0) + \omega_0 (t - t_0) \).
Damped Oscillator

A typical response for the unperturbed system is

\[ \theta(t) \]

\[ I(t) \]

\[ q(t) \]

\[ p(t) \]

Figure: Linearized Oscillator Response
The transformation $\mathcal{H}(\theta, I, t) \rightarrow \mathcal{K}(\hat{\theta}, \hat{I}, t)$ is defined implicitly through the pair of generating functions $\mathcal{W}(\hat{\theta}, \hat{I}, t)$ and $\mathcal{V}(\hat{\theta}, \hat{I}, t)$ which are themselves derived from the solution to the homological differential equations in either of the two forms

$$\frac{\partial \mathcal{W}_{m,n}}{\partial t} + \omega_0 \frac{\partial \mathcal{W}_{m,n}}{\partial \theta} = \mathcal{Q}_{m,n} - \mathcal{K}_{m,n}$$

$$\frac{\partial \mathcal{V}_{m,n}}{\partial t} + \omega_0 \frac{\partial \mathcal{V}_{m,n}}{\partial \theta} = \mathcal{P}_{m,n} - \mathcal{K}_{m,n}$$

where the terms $\mathcal{K}_{m,n}$ constitute the transformed Hamiltonian function and the terms $\mathcal{Q}_{m,n}$ and $\mathcal{P}_{m,n}$ are known prior to solving the homological equations.
Damped Oscillator

(a) 0\textsuperscript{th}-Order Solution

(b) 2\textsuperscript{nd}-Order Solution
Damped Oscillator

(c) $4^{th}$-Order Solution

(d) $6^{th}$-Order Solution
Damped Oscillator

(a) 0\textsuperscript{th}-Order Solution

(b) 2\textsuperscript{nd}-Order Solution
Damped Oscillator

(c) 4\textsuperscript{th}-Order Solution

(d) 6\textsuperscript{th}-Order Solution
A conservative approach to achieving this controlled behavior is to eliminate all the secular variations in the system

$$\mathcal{K}_c = \mathcal{K}_{0,0} = \omega_0 \hat{I}$$

The control law is defined at each order in the expansion by

$$\mathcal{K}_{c,i,j} - \mathcal{K}_{i,j} = -\frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L}_i \mathcal{W}_{u,i+1,j} \mathcal{H}_{0,0} + \mathcal{Q}_{u,i,j}$$

$$+ \mathcal{U}_{i,j}^{(0,0)} + \mathcal{Q}_{c,i,j}$$

Consider the case where the control generating functions are zero, $\mathcal{W}_u = \mathcal{V}_u = 0$ and the control law reduces to

$$\mathcal{U}_{i,j}^{(0,0)} = \mathcal{K}_{i,j} - \mathcal{Q}_{c,i,j} \quad i + j \neq 0$$
Damped Oscillator

(a) $0^{th}$-Order Solution

(b) $2^{nd}$-Order Solution
Damped Oscillator

(c) 4\textsuperscript{th}-Order Solution

(d) 6\textsuperscript{th}-Order Solution
Damped Oscillator

(a) $0^{th}$-Order Solution

(b) $2^{nd}$-Order Solution
Damped Oscillator

(c) 4\textsuperscript{th}-Order Solution

(d) 6\textsuperscript{th}-Order Solution
Damped Oscillator

(a) $0^{\text{th}}$-Order Solution

(b) $2^{\text{nd}}$-Order Solution

(c) $4^{\text{th}}$-Order Solution

(d) $6^{\text{th}}$-Order Solution